

POLYNOMIAL REPRESENTATIONS AND CATEGORIFICATIONS OF FOCK SPACE

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ABSTRACT. The rings of symmetric polynomials form an inverse system whose limit, the ring of symmetric functions, is the model for the bosonic Fock space representation of the affine Lie algebra. We categorify this construction by considering an inverse limit of categories of polynomial representation of general linear groups. We show that this limit naturally carries an action of the affine Lie algebra (in the sense of Rouquier), thereby obtaining a family of categorifications of the bosonic Fock space representation.

1. INTRODUCTION

A basic object at the intersection of representation theory and algebraic combinatorics is the ring of symmetric functions in infinitely many variables. This ring is constructed via the limit of the inverse system

$$\cdots \rightarrow B_n \rightarrow B_{n-1} \rightarrow \cdots$$

where $B_n = \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ is the symmetric polynomials in n indeterminates, and $B_n \rightarrow B_{n-1}$ is the map obtained by setting $x_n = 0$. The ring B of symmetric functions is then defined as the subring of $\varprojlim B_n$ consisting of elements of finite degree.

The algebra B possesses a striking array of symmetries, which are related to many classical structures. In this paper we focus on an affine Lie algebra action which realizes B as the (bosonic) Fock space representation of $\widehat{\mathfrak{sl}}_n$. Our present purpose is to categorify the limit construction of B along with the action of the affine Lie algebra.

To describe this idea in more detail, fix an algebraically closed field \mathbb{F} of characteristic $p \geq 0$ and let \mathfrak{g} be the complex Kac-Moody algebra $\widehat{\mathfrak{sl}}_p$, or \mathfrak{sl}_∞ when $p = 0$. Consider the category \mathcal{M}_n of polynomial representations of $\mathrm{GL}_n(\mathbb{F})$. It is well known that \mathcal{M}_n categorifies B_n , i.e. the representation ring of \mathcal{M}_n is precisely the symmetric polynomials in n variables. Moreover, the categories \mathcal{M}_n form a direct system

$$\cdots \rightarrow \mathcal{M}_n \rightarrow \mathcal{M}_{n-1} \rightarrow \cdots$$

where $\mathcal{M}_n \rightarrow \mathcal{M}_{n-1}$ is the functor $V \mapsto V^{\mathrm{GL}_1}$, i.e. the invariants with respect to the GL_1 in the lower right-hand corner commuting with the standard $\mathrm{GL}_{n-1} \subset \mathrm{GL}_n$. (This functor categorifies the map $B_n \rightarrow B_{n-1}$.)

From this direct system we form a category $\varprojlim \mathcal{M}_n$, and by imposing natural finite-ness conditions we define a subcategory $\mathcal{M} \subset \varprojlim \mathcal{M}_n$ (Section 3). \mathcal{M} is a tensor category, whose Grothendieck group is naturally isomorphic to the ring B . Our main result is that there is an action of \mathfrak{g} on \mathcal{M} (in the sense of Rouquier) which categorifies the Fock space representation of B (Theorem 5.12). This means that besides defining a family of endofunctors on \mathcal{M} which give an integrable representation of \mathfrak{g} on the Grothendieck

group $K(\mathcal{M})$, we also describe the additional data of a degenerate affine Hecke algebra action on a certain sum of these functors (see Section 4 for a precise definition).

As a consequence of Chuang-Rouquier theory we obtain derived equivalences between certain blocks of \mathcal{M} (Corollary 5.13). Using results of Brundan and Kleshchev we recover the crystal of the Fock space naturally from our construction (Corollary 5.16). The vertices of the corresponding crystal graph are the simple objects in \mathcal{M} .

The construction of a “strong” categorification on a limit of categories is novel, and, we believe, interesting in its own right. Moreover it seems to be adaptable to other settings, such as representations of quantum groups. Some of the other constructions appearing in this paper have their origin in the earlier works on categorification and representation theory, such as [BFK]. Our methods are also influenced by the work of Chuang and Rouquier [CR]. They studied \mathfrak{sl}_2 -categorifications on $\text{Rep}(\text{GL}_n)$, the category of *rational* representations of GL_n . Our work can be viewed as a sort of limit of their construction, which allows us to obtain the Fock space representation, rather than exterior powers of the standard representation.

In sequels to this work, we develop this theory from the point of view of the strict polynomial functors of MacDonalld [M] and Friedlander-Suslin [FS], and relate these results to Khovanov’s Heisenberg categorification and Schur-Weyl duality [HTY],[HY]. Finally, we mention that a different \mathfrak{g} -categorification of the Fock space was constructed in [Sh] using the category \mathcal{O} for the double affine Hecke algebra. Moreover, Stroppel and Webster recently used cyclotomic \mathfrak{q} -Schur algebras to categorify higher level quantum Fock space [SW].

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2. PRELIMINARIES

In this section we set some notational conventions. Section 2.1 reviews some modular representation theory of GL_n and introduces the functors R_i which are used frequently throughout this paper. In Section 2.2 we recall the definition of the Fock space representation of \mathfrak{g} .

2.1. The general linear group. Let \mathbb{F} be an algebraically closed field of characteristic $p \neq 2$. (We exclude the $p = 2$ case for ease of exposition; modulo some technical complications all of our constructions carry over to that case.) Let \mathbb{F}^∞ be the infinite dimensional vector space with distinguished basis $\{v_1, v_2, \dots\}$. We fix the standard flag $\mathbb{F}^0 \subset \mathbb{F}^1 \subset \mathbb{F}^2 \subset \dots$ of vector spaces, where \mathbb{F}^n is the span of $\{v_1, \dots, v_n\}$.

Let G_n denote the rank n general linear group:

$$G_n = \text{GL}(\mathbb{F}^n).$$

Set $T_n \subset G_n$ to be the maximal torus consisting of diagonal matrices, and let B_n be the Borel subgroup of upper triangular matrices and let B_n^- be the opposite Borel subgroup.

We denote by $X(T_n)$ the character group of T_n . We identify $X(T_n)$ with n -tuples of integers: an n -tuple (k_1, \dots, k_n) gives rise to the character $\text{diag}(t_1, \dots, t_n) \mapsto t_1^{k_1} \cdots t_n^{k_n}$. We assume the roots appear in B_n to be positive. Let $X(T_n)_+$ be the cone of dominant

weights with respect to the positive roots; elements of $X(\mathbb{T}_n)_+$ are of the form $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$. For $\lambda \in X(\mathbb{T}_n)_+$, we set $|\lambda| = \lambda_1 + \dots + \lambda_n$.

Let $\text{Rep}(\mathbb{G}_n)$ denote the category of rational representations of \mathbb{G}_n . A rational representation of \mathbb{G}_n is said to be *polynomial* if all its matrix coefficients can be extended to polynomials on M_n . Denote by \mathcal{M}_n the category of finite dimensional polynomial representations of \mathbb{G}_n .

Let $Z_n \subset \mathbb{G}_n$ be the center of \mathbb{G}_n , which consists of scalar matrices. Given a representation V_n of \mathbb{G}_n , we can decompose it into weight spaces with respect to the action of Z_n :

$$V_n = \bigoplus_{k \in \mathbb{Z}} V_n(k),$$

where

$$V_n(k) = \{v \in V_n : z.v = z^k v \text{ for all } z \in Z_n\}.$$

The representation V_n is said to be of *degree* k if $V_n = V_n(k)$. If V_n is of degree k for some k , then we say V_n is *homogeneous*. A polynomial representation of \mathbb{G}_n is a direct sum of homogeneous representations of non-negative degrees, and so, in particular, simple modules are homogeneous. Let $\mathcal{M}_n(k)$ be the category of polynomial representations of \mathbb{G}_n of degree k .

For all $n \geq 1$, we embed $\mathbb{G}_{n-1} \subset \mathbb{G}_n$ as the automorphisms fixing v_n . Given $V \in \text{Rep}(\mathbb{G}_n)$, we denote its restriction to \mathbb{G}_{n-1} by $V|_{\mathbb{G}_{n-1}}$. Let

$$R : \mathcal{M}_n \rightarrow \mathcal{M}_{n-1}$$

be the functor assigning to a representation V of \mathbb{G}_n its restriction to \mathbb{G}_{n-1} :

$$R(V) = V|_{\mathbb{G}_{n-1}}.$$

Let V be a representation of \mathbb{G}_n . There is an action of \mathbb{G}_1 on $R(V)$, coming from the automorphisms which fix all vectors v_i except v_n . This copy of \mathbb{G}_1 commutes with $\mathbb{G}_{n-1} \subset \mathbb{G}_n$, i.e. $\mathbb{G}_{n-1} \times \mathbb{G}_1 \subset \mathbb{G}_n$. Since \mathbb{G}_1 acts semi-simply, we can decompose the functor

$$(1) \quad R = \bigoplus_{i \in \mathbb{Z}} R_i,$$

corresponding to the weight spaces of \mathbb{G}_1 . In particular, $R_0(V) = R(V)^{\mathbb{G}_1}$, the \mathbb{G}_1 -invariant vectors.

For $\lambda \in X(\mathbb{T}_n)$ we set

$$H^0(\lambda) = \text{ind}_{B_n^-}^{\mathbb{G}_n}(\mathbb{F}_\lambda),$$

where \mathbb{F}_λ is the one-dimensional B_n^- -module of weight λ . For $\lambda \in X(\mathbb{T}_n)_+$ set

$$L_n(\lambda) = \text{soch}^0(\lambda),$$

the maximal semi-simple submodule of $H^0(\lambda)$.

Note that $L_n(\lambda)$ is a polynomial representation if and only if $\lambda_i \geq 0$ for all i , where $\lambda = (\lambda_1, \dots, \lambda_n)$. In general, the representation $L_n(\lambda)$ is a simple module of highest weight λ ; this implies $L_n(\lambda) \in \mathcal{M}_n(|\lambda|)$ when λ is a highest weight of a polynomial representation. Moreover,

Proposition 2.1 ([J], Corollary II.2.7). *The representations $L_n(\lambda)$ with $\lambda \in X(\mathbb{T}_n)_+$ are a system of representatives for the isomorphism classes of simple rational \mathbb{G}_n -modules.*

Recall that the Weyl group of G_n is isomorphic to \mathfrak{S}_n , the symmetric group on n letters, which acts on $X(T_n)$ by place permutations. Let w_0 be the longest element of the Weyl group.

Definition 2.2. For $\lambda \in X(T_n)_+$ the Weyl module $V_n(\lambda) \in \mathcal{M}_n(|\lambda|)$ of highest weight λ is given by

$$V_n(\lambda) = H^0(-w_0\lambda)^*.$$

The G_n -module $V_n(\lambda)$ is generated by a B_n -stable line of weight λ . Moreover, $V_n(\lambda)$ is universal with respect to this property, i.e. any G_n -module generated by a B_n -stable line of weight λ is a homomorphic image of $V_n(\lambda)$. In particular,

$$(2) \quad V_n(\lambda)/\text{rad}_G V_n(\lambda) \cong L_n(\lambda).$$

Definition 2.3 ([J], §II.4.19). An ascending chain $0 = V_0 \subset V_1 \subset \dots$ of submodules of a G_n -module $V \in \text{Rep}(G_n)$ is called a Weyl filtration if $V = \bigcup V_i$ and each factor V_i/V_{i-1} is isomorphic to some Weyl module

Let \mathfrak{g}_n be the Lie algebra of G_n ; this is the Lie algebra of $n \times n$ matrices over \mathbb{F} . The matrix units are denoted by x_{ij} . Let $U(\mathfrak{g}_n)$ be the universal enveloping algebra of \mathfrak{g}_n with center $Z(\mathfrak{g}_n)$. The Casimir operator of $U(\mathfrak{g}_n)$ is

$$(3) \quad C_n = \sum_{i \neq j} x_{ij}x_{ji} + \sum_{\ell=1}^n x_{\ell\ell}^2 \in Z(\mathfrak{g}_n).$$

For $\lambda = (\lambda_1, \dots, \lambda_n) \in X(T_n)$, define the scalar

$$(4) \quad c_n(\lambda) = \sum_{i=1}^n (n - 2i + 1)\lambda_i + \lambda_i^2.$$

It is an elementary computation that on any module V which is generated by a vector of weight λ , C_n acts by the scalar $c_n(\lambda)$. In particular,

Lemma 2.4. For $\lambda \in X(T)_+$, the Casimir operator C_n acts on $V_n(\lambda)$ and $L_n(\lambda)$ by the scalar $c_n(\lambda)$.

2.2. Kac-Moody algebras in type A. Let \mathfrak{g} denote the following Kac-Moody algebra (over \mathbb{C}):

$$\mathfrak{g} = \begin{cases} \mathfrak{sl}_\infty & \text{if } p = 0 \\ \widehat{\mathfrak{sl}}_p & \text{if } p > 0 \end{cases}$$

By definition, the Kac-Moody algebra \mathfrak{sl}_∞ is associated to the Dynkin diagram:

$$\dots \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots$$

The Kac-Moody algebra $\widehat{\mathfrak{sl}}_p$ is associated to the diagram with p nodes:

$$\bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet$$

For the precise relations defining \mathfrak{g} see [K]. The Lie algebra \mathfrak{g} has standard Chevalley generators $\{e_i, f_i\}_{i \in \mathbb{Z}/p\mathbb{Z}}$. Here, and throughout, we identify $\mathbb{Z}/p\mathbb{Z}$ with the prime subfield of \mathbb{F} .

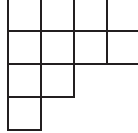
Set $\mathfrak{h}_i = [e_i, f_i]$. We let Q denote the root lattice and P the weight lattice of \mathfrak{g} . The cone of dominant weights is denoted P_+ with generators the fundamental weights

$\{\omega_i : i \in \mathbb{Z}/p\mathbb{Z}\}$. Let $\mathcal{L}(\omega)$ be the irreducible \mathfrak{g} -module of highest weight $\omega \in P_+$. In particular, the *basic representation* is $\mathcal{L}(\omega_0)$.

Of central importance to us is the *Fock space* representation of \mathfrak{g} , which we now define. Let Λ be the set of all partitions. As a vector space \mathcal{F} has basis indexed by partitions:

$$\mathcal{F} = \bigoplus_{\lambda \in \Lambda} \mathbb{C}v_\lambda.$$

We identify partitions with their Young diagram (using English notation). For example the partition $(4, 4, 2, 1)$ corresponds to the diagram



The *content* of a box in position (k, l) is the integer $l - k \in \mathbb{Z}/p\mathbb{Z}$. Given $\mu, \lambda \in \Lambda$, we write

$$\mu \longrightarrow \lambda$$

if λ can be obtained from μ by adding some box. If the arrow is labelled i then λ is obtained from μ by adding a box of content i (an *i*-box, for short). For instance, if $\mu = (2)$, $\lambda = (2, 1)$, and $m = 3$ then

$$\mu \xrightarrow{2} \lambda.$$

An *i*-box of λ is *addable* (resp. *removable*) if it can be added to (resp. removed from) λ to obtain another partition. Let $m_i(\lambda)$ denote the number of *i*-boxes of λ . Define

$$(5) \quad n_i(\lambda) = m_{i-1}(\lambda) + m_{i+1}(\lambda) - 2m_i(\lambda) + \delta_{i0},$$

where δ_{i0} is the Kronecker delta.

The action of \mathfrak{g} on \mathcal{F} is given on Chevalley generators by the following formulas:

$$e_i \cdot v_\lambda = \sum v_\mu$$

the sum over all μ such that $\mu \xrightarrow{i} \lambda$, and

$$f_i \cdot v_\lambda = \sum v_\mu$$

the sum over all μ such that $\lambda \xrightarrow{i} \mu$. These equations define an integral representation of \mathfrak{g} (see e.g. [LLT]).

Note that v_\emptyset is a highest weight vector of highest weight ω_0 . We note also that the standard basis of \mathcal{F} is a weight basis, any weight vector v_λ is of weight

$$(6) \quad \omega_0 - \sum_i m_i \alpha_i.$$

Indeed, from the above formulas one obtains:

$$h_i \cdot v_\lambda = n_i(\lambda)v_\lambda.$$

3. THE CATEGORY \mathcal{M}

In this section we define our main object of study, the category \mathcal{M} .

3.1. The stable category. In this section we describe a stable category constructed from polynomial representations of general linear groups. Recall the functor $R_0 : \mathcal{M}_{n+1} \rightarrow \mathcal{M}_n$, which is defined by taking the invariants with respect to G_1 :

$$R_0(V_{n+1}) = V_{n+1}^{G_1}.$$

Since R_0 preserves degree, it also defines a functor $R_0 : \mathcal{M}_{n+1}(k) \rightarrow \mathcal{M}_n(k)$.

Proposition 3.1 (Theorem 4.3.6, [Mar]). *For $n \geq k$, $R_0 : \mathcal{M}_{n+1}(k) \rightarrow \mathcal{M}_n(k)$ is an equivalence of categories.*

We have a diagram of functors:

$$\dots \xrightarrow{R_0} \mathcal{M}_2 \xrightarrow{R_0} \mathcal{M}_1 \xrightarrow{R_0} \mathcal{M}_0.$$

The category \mathcal{M} will be constructed by first defining the limit category of this diagram, and then taking the subcategory of compact objects.

Definition 3.2. *Let $\widetilde{\mathcal{M}}$ be the category whose objects are sequences*

$$V = (V_n, \alpha_n)_{n=0}^\infty$$

where $V_n \in \mathcal{M}_n$ and

$$\alpha_n : R_0(V_{n+1}) \rightarrow V_n$$

is an isomorphism of G_n -modules (by convention G_0 is the trivial group).

For objects $V = (V_n, \alpha_n)_{n=0}^\infty$ and $W = (W_n, \beta_n)_{n=0}^\infty$ in $\widetilde{\mathcal{M}}$, the space of morphisms $\text{Hom}_{\widetilde{\mathcal{M}}}(V, W)$ is defined as the limit

$$\varprojlim \text{Hom}_{G_n}(V_n, W_n).$$

Here we are using the maps

$$\theta_n^{n+1} : \text{Hom}_{G_{n+1}}(V_{n+1}, W_{n+1}) \rightarrow \text{Hom}_{G_n}(V_n, W_n)$$

given by $f_{n+1} \mapsto \beta_n \circ R_0(f_{n+1}) \circ \alpha_n^{-1}$.

By definition, a morphism $f \in \text{Hom}_{\widetilde{\mathcal{M}}}(V, W)$ is a sequence $(f_n)_{n=0}^\infty$, where

$$f_n \in \text{Hom}_{G_n}(V_n, W_n)$$

and the following diagram commutes

$$\begin{array}{ccc} R_0(V_{n+1}) & \xrightarrow{R_0(f_{n+1})} & R_0(W_{n+1}) \\ \downarrow \alpha_n & & \downarrow \beta_n \\ V_n & \xrightarrow{f_n} & W_n \end{array}$$

for $n \gg 0$. Two morphisms $f, g : V \rightarrow W$ are equal if for $n \gg 0$, $f_n = g_n$.

When there is no cause for confusion, we abbreviate an object $(V_n, \alpha_n)_{n=0}^\infty$ in $\widetilde{\mathcal{M}}$ by (V_n, α_n) , or sometimes simply (V_n) when the maps α_n are understood from context. Similarly a morphism $(f_n)_{n=0}^\infty$ will be abbreviated as (f_n) .

If $V_n, W_n \in \mathcal{M}_n$, then canonically $R_0(V_n \otimes W_n) \cong R_0(V_n) \otimes R_0(W_n)$. Therefore there is a monoidal structure on $\widetilde{\mathcal{M}}$ described as follows: for $V = (V_n, \alpha_n)$ and $W = (W_n, \beta_n)$ objects in $\widetilde{\mathcal{M}}$, set $V \otimes W = (V_n \otimes W_n, \alpha_n \otimes \beta_n)$. The following lemma is straight-forward.

Lemma 3.3. *$\widetilde{\mathcal{M}}$ is a symmetric tensor category.*

Example 3.4. *Here are some naturally occurring objects in $\widetilde{\mathcal{M}}$.*

- (1) Let $\mathbf{1}_n$ be the trivial one dimensional representation of G_n . Clearly, $R_0(\mathbf{1}_{n+1}) \cong \mathbf{1}_n$, so the trivial representations glue together to the unit object $\mathbf{1} = (\mathbf{1}_n)$ in the symmetric tensor category $\widetilde{\mathcal{M}}$.
- (2) Consider the standard representation of G_n on \mathbb{F}^n . Then $R_0(\mathbb{F}^{n+1})$ is precisely the span of v_1, \dots, v_n , so using the obvious morphisms $\iota_n : R_0(\mathbb{F}^{n+1}) \rightarrow \mathbb{F}^n$, we obtain the “standard” object $\mathbf{St} = (\mathbb{F}^n, \iota_n) \in \widetilde{\mathcal{M}}$.
- (3) Fix a nonnegative integer r , and consider the tensor product representation of G_n on $\bigotimes^r \mathbb{F}^n$. Then, as above, there are obvious morphisms $R_0(\bigotimes^r \mathbb{F}^{n+1}) \rightarrow \bigotimes^r \mathbb{F}^n$, which glue together to an object which is canonically isomorphic to $\bigotimes^r \mathbf{St}$. One can similarly define objects $S^r(\mathbf{St})$ and $\bigwedge^r \mathbf{St}$.
- (4) In contrast to the tensor algebra and symmetric algebra of \mathbb{F}^n , the exterior algebra is finite dimensional so $\bigwedge \mathbb{F}^n \in \mathcal{M}_n$. Therefore, from isomorphisms $R_0(\bigwedge \mathbb{F}^{n+1}) \rightarrow \bigwedge \mathbb{F}^n$, we can define an object which we denote $\bigwedge \mathbf{St} = (\bigwedge \mathbb{F}^n) \in \widetilde{\mathcal{M}}$.

Definition 3.5. An object $V = (V_n) \in \widetilde{\mathcal{M}}$ is of degree k if for every n , V_n is of degree k . If V is of degree k for some k , then we say V is homogeneous. Let $\mathcal{M}(k)$ denote the subcategory of $\widetilde{\mathcal{M}}$ consisting of objects of degree k .

For an object $V = (V_n) \in \widetilde{\mathcal{M}}$, let $V(k) = (V_n(k))$. We have the (possibly infinite) direct sum in $\widetilde{\mathcal{M}}$:

$$(7) \quad V = \bigoplus V(k).$$

An object $V \in \widetilde{\mathcal{M}}$ is *compact* if the direct sum above is finite. This is equivalent to the usual notion of compactness from category theory, namely that V commutes with coproducts. For instance, in the example above (1)-(3) are compact objects, while $\bigwedge \mathbf{St}$ is not. We thus arrive at a different realization of the category of polynomial functors.

Definition 3.6. The category \mathcal{M} is the full subcategory of $\widetilde{\mathcal{M}}$ consisting of compact objects, i.e. those objects where the sum (7) is finite.

Note that \mathcal{M} is the direct sum of categories:

$$\mathcal{M} = \bigoplus_{k=0}^{\infty} \mathcal{M}(k),$$

and that \mathcal{M} is also a tensor category. Let Ψ_n be the projection functor from \mathcal{M} to \mathcal{M}_n , and let $\Psi_n(k)$ denote its restriction to $\mathcal{M}(k)$. By Proposition 3.1, we have:

Proposition 3.7. For $n \geq k$, $\Psi_n(k) : \mathcal{M}(k) \rightarrow \mathcal{M}_n(k)$ is an equivalence of categories.

4. PROPERTIES OF \mathcal{M}

In this section we undertake a thorough study of the category \mathcal{M} in preparation for the categorification theorem we prove in the next section. In Section 4.1 we use polynomial induction functors ([BK1]) to construct an inverse functor to R_0 . In Section 4.2 we study some standard objects in \mathcal{M} . In Section 4.3 we introduce the functors E and F and decompose them into subfunctors using endomorphisms $X \in \text{End}(E)$ and $Y \in \text{End}(F)$.

4.1. Polynomial induction. We recall the notion of polynomial induction due to Brundan and Kleshchev [BK1], and give self-contained proofs of a few of their results.

Set $M_{m,n} = \text{Hom}(\mathbb{F}^n, \mathbb{F}^m)$, and for convenience let $M_n = M_{n,n}$. Let $\mathcal{O}(M_{m,n})$ be the algebra of polynomials on $M_{m,n}$. There exists a natural action of $G_m \times G_n$ on $M_{m,n}$ by

$$(g_1, g_2) \cdot A = g_1 A g_2^\dagger.$$

Then $G_m \times G_n$ acts on $\mathcal{O}(M_{m,n})$ by

$$((g_1, g_2) \cdot f)(A) = f(g_1^\dagger A g_2).$$

Definition 4.1. Let I_0 be the functor from $\text{Rep}(G_n)$ to $\text{Rep}(G_{n+1})$ given by

$$V_n \mapsto (V_n \otimes \mathcal{O}(M_{n,n+1}))^{G_n}.$$

Here the invariants are taken with respect to the tensor product action of G_n on $V_n \otimes \mathcal{O}(M_{n,n+1})$. This action commutes with the action of G_{n+1} on $\mathcal{O}(M_{n,n+1})$ by right translation, thus resulting in a G_{n+1} -module. By Proposition A.3 in [J], $I_0(V_n)$ is a polynomial representation.

Now let $O_P : \text{Rep}(G_n) \rightarrow \text{Rep}(G_n)$ be the functor assigning to a representation its maximal polynomial submodule. Define $\text{Ind}_n^{n+1} : \text{Rep}(G_n) \rightarrow \text{Rep}(G_{n+1})$ by

$$\text{Ind}_n^{n+1}(V_n) = (V_n \otimes \mathcal{O}(G_{n+1}))^{G_n},$$

where $\mathcal{O}(G_{n+1})$ denotes the algebra of regular functions on G_{n+1} . Here, G_{n+1} acts on $\mathcal{O}(G_{n+1})$ by right translation, while G_n acts by left translation.

Definition 4.2. The polynomial induction functor, denoted Pind_n^{n+1} , is the composition of functors $O_P \circ \text{Ind}_n^{n+1}$.

By Equation (1) in Section A.18 in [J], $O_P(\mathcal{O}(G_{n+1})) = \mathcal{O}(M_{n+1})$. From this we obtain another formulation of polynomial induction:

$$\text{Pind}_n^{n+1}(V_n) \simeq (V_n \otimes \mathcal{O}(M_{n+1}))^{G_n}.$$

Lemma 4.3. There exists a natural G_{n+1} -isomorphism

$$\text{Pind}_n^{n+1}(V_n) \simeq S(\mathbb{F}^{n+1}) \otimes I_0(V_n).$$

Here $S(\mathbb{F}^{n+1})$ denotes the symmetric algebra of \mathbb{F}^{n+1} , endowed with the standard G_{n+1} action induced from the natural representation, and G_{n+1} acts on $S(\mathbb{F}^{n+1}) \otimes I_0(V_n)$ by the tensor product action.

Proof. As a $G_n \times G_{n+1}$ -module $M_{n+1}^* \cong \mathbb{F}^{n+1} \oplus M_{n,n+1}^*$. Here M_{n+1}^* and $M_{n,n+1}^*$ are the linear duals of M_{n+1} , and $M_{n,n+1}$ and \mathbb{F}^{n+1} is considered a $G_n \times G_{n+1}$ -module with G_n acting trivially. Hence as a $G_n \times G_{n+1}$ -module $\mathcal{O}(M_{n+1}) \cong S(\mathbb{F}^{n+1}) \otimes \mathcal{O}(M_{n,n+1})$, where $S(\mathbb{F}^{n+1})$ is trivial as a G_n -module. The result follows. \square

Lemma 4.4. I_0 defines a functor $\mathcal{M}_n \rightarrow \mathcal{M}_{n+1}$, and $R_0 : \mathcal{M}_{n+1} \rightarrow \mathcal{M}_n$ is left adjoint to $I_0 : \mathcal{M}_n \rightarrow \mathcal{M}_{n+1}$.

Proof. It suffices to show that for all $V_{n+1} \in \mathcal{M}_{n+1}$ and $W_n \in \mathcal{M}_n$

$$\text{Hom}_{G_n}(R_0(V_{n+1}), W_n) \cong \text{Hom}_{G_{n+1}}(V_{n+1}, I_0(W_n)).$$

Since the functors R_0 and I_0 preserve degree, we can assume that V_{n+1} and W_n are of the same degree k . In this case we have

$$\begin{aligned} \mathrm{Hom}_{G_n}(R_0(V_{n+1}), W_n) &\cong \mathrm{Hom}_{G_n}(R(V_{n+1}), W_n) \\ &\cong \mathrm{Hom}_{G_{n+1}}(V_{n+1}, \mathrm{Pind}_n^{n+1}W_n) \\ &\cong \mathrm{Hom}_{G_{n+1}}(V_{n+1}, I_0(W_n)). \end{aligned}$$

Note that the second isomorphism follows from Frobenius reciprocity, and the last one follows from Lemma 4.3 by degree considerations. \square

Let $I_{n,n+1}$ be the $n \times n+1$ matrix with 1 in entry (i, i) for $1 \leq i \leq n$ and 0 elsewhere.

Corollary 4.5. *For $n \geq k$,*

$$I_0 : \mathcal{M}_n(k) \rightarrow \mathcal{M}_{n+1}(k)$$

is an inverse functor, up to isomorphism, of R_0 . Moreover, the isomorphism $R_0 I_0 \rightarrow \mathrm{Id}$ is given by evaluation at $I_{n,n+1}$.

Proof. Recall that an adjoint of an equivalence is necessarily isomorphic to an inverse functor. Therefore, by Proposition 3.1 and Lemma 4.4, the first statement of the corollary follows. This implies that the counit $R_0 I_0 \rightarrow \mathrm{Id}$ is an isomorphism. It remains to show that it is given by evaluation at $I_{n,n+1}$.

The counit evaluated at $V_n \in \mathcal{M}_n$ is the morphism $R_0 I_0(V_n) \rightarrow V_n$ obtained by passing the identity morphism of $I_0(V_n)$ through the chain of isomorphisms appearing in the proof of Lemma 4.4. Viewing elements of $\mathrm{Pind}_n^{n+1}(V_n)$ as functions from M_{n+1} to V_n , and elements of $I_0(V_n)$ as functions from $M_{n,n+1}$ to V_n , the corollary follows from the following two observations. Firstly, the isomorphism $\mathrm{Hom}_{G_{n+1}}(V_{n+1}, I_0(W_n)) \cong \mathrm{Hom}_{G_{n+1}}(V_{n+1}, \mathrm{Pind}_n^{n+1}W_n)$ maps a function $f : V_{n+1} \rightarrow I_0(W_n)$ to $(v \mapsto f(v) \circ \mathrm{pr})$, where pr is the projection from $M_{n+1} \rightarrow M_{n,n+1}$. Secondly, the Frobenius reciprocity isomorphism $\mathrm{Hom}_{G_{n+1}}(V_{n+1}, \mathrm{Pind}_n^{n+1}W_n) \cong \mathrm{Hom}_{G_n}(R(V_{n+1}), W_n)$ maps $f : V_{n+1} \rightarrow \mathrm{Pind}_n^{n+1}W_n$ to $(v \mapsto f(v)(I_{n+1}))$. \square

Lemma 4.6. *Let $n \geq k$ and $\lambda \in X(T_n)_+$ such that $|\lambda| = k$. Then $R_0(V_{n+1}(\lambda)) \simeq V_n(\lambda)$ and $I_0(V_n(\lambda)) \simeq V_{n+1}(\lambda)$.*

Proof. The first claim follows by the following series of equalities of characters:

$$\begin{aligned} \mathrm{char}_{\mathbb{F}}(R_0(V_{n+1}(\lambda))) &= \mathrm{char}_{\mathbb{C}}(R_0(V_{n+1}(\lambda))) \\ &= \mathrm{char}_{\mathbb{C}}(V_n(\lambda)) \\ &= \mathrm{char}_{\mathbb{F}}(V_n(\lambda)). \end{aligned}$$

The first equality is a consequence of that fact that $R_0(V_{n+1}(\lambda))$ admits a Weyl filtration (Theorem A.1). The second equality follows by the classical branching rules for the general linear groups, and the last one by definition of Weyl modules.

The second isomorphism in the statement of the lemma follows from the first by Corollary 4.5. \square

Given $(V_n, \alpha_n) \in \mathcal{M}$, let $\alpha_n^\vee : V_{n+1} \rightarrow I_0(V_n)$ be the morphism induced from $\alpha_n : R_0(V_{n+1}) \rightarrow V_n$ by Lemma 4.4. Then α_n^\vee is an isomorphism, and Lemma 4.4 and Corollary 4.5 imply the following:

Lemma 4.7. *We have the following commutative diagram:*

$$\begin{array}{ccc} \mathbf{R}_0(V_{n+1}) & \xrightarrow{\alpha_n^\vee} & \mathbf{R}_0 \circ \mathbf{I}_0(V_n) \\ & \searrow \alpha_n & \downarrow \text{ev}_{\mathbf{I}_0, n+1} \\ & & V_n \end{array}$$

4.2. **Objects in \mathcal{M} .** We parameterize the Weyl (i.e. standard) and simple objects in \mathcal{M} .

4.2.1. *The projection functor.* By Proposition 3.7 we have the following equivalence of categories:

$$(8) \quad \Psi : \bigoplus_{k=0}^{\infty} \mathcal{M}(k) \rightarrow \bigoplus_{k=0}^{\infty} \mathcal{M}_k(k)$$

which is defined by taking the direct sum of projection functors $\Psi_k(k)$. We now describe the inverse functor to Ψ .

Definition 4.8. *To a representation $V_k \in \mathcal{M}_k(k)$, we associate an object $\Psi^{-1}(V_k) \in \mathcal{M}$ as follows: $\Psi^{-1}(V_k) = (V_n)_{n=0}^{\infty}$ where for $n \geq k$*

$$V_n = \mathbf{I}_0^{n-k}(V_k)$$

and for $n < k$

$$V_n = \mathbf{R}_0^{k-n}(V_k).$$

Since \mathbf{I}_0 and \mathbf{R}_0 are inverse functors to each other for $n \geq k$ (Corollary 4.5), this gives a well-defined object in \mathcal{M} . The same formulas apply to morphisms as well; we thus obtain a functor $\Psi^{-1} : \mathcal{M}_k \rightarrow \mathcal{M}$. By taking direct sum we obtain the functor

$$\Psi^{-1} : \bigoplus_{k=0}^{\infty} \mathcal{M}_k(k) \rightarrow \mathcal{M}.$$

The next proposition, which follows directly from the fact that \mathbf{I}_0 and \mathbf{R}_0 are inverse functors up to isomorphism, justifies our choice of notation.

Proposition 4.9. *The functor Ψ^{-1} is inverse to Ψ .*

4.2.2. *Simple objects in \mathcal{M} .* Recall that Λ is the set of all partitions. Let Λ_k be the set of partitions of k .

Proposition 4.10. *The simple objects of \mathcal{M} are in canonical one-to-one correspondence with Λ .*

Proof. By Proposition 2.1, the simple objects of $\mathcal{M}_k(k)$ are, up to isomorphism, precisely

$$\{L_k(\lambda) : \lambda \in \Lambda_k\}$$

where we identify the partition $\lambda \in \Lambda_k$ with the weight $\lambda \in X(\mathbb{T}_k)_+$. It follows that the simple objects on the right hand side of (8) are, up to isomorphism, given by

$$\{L_{|\lambda|}(\lambda) : \lambda \in \Lambda\}.$$

Since Ψ is an equivalence the result follows. \square

Definition 4.16. Define a functor $E : \mathcal{M} \rightarrow \mathcal{M}$ as follows. Let $V = (V_n, \alpha_n) \in \mathcal{M}$. Then $E(V) = (E(V)_n, \tilde{\alpha}_n)$, where

$$E(V)_n = R_1(V_{n+1})$$

and

$$\tilde{\alpha}_n = R_1(\alpha_{n+1}) \circ s_{n+1}^{-1}.$$

Given a morphism

$$f = (f_n) : (V_n, \alpha_n) \rightarrow (W_n, \beta_n),$$

set $E(f) = (E(f)_n)$, where

$$E(f)_n = R_1(f_{n+1}).$$

Lemma 4.17. The functor $E : \mathcal{M} \rightarrow \mathcal{M}$ is well-defined.

Proof. Let $V \in \mathcal{M}$. It is clear that $E(V) \in \mathcal{M}$. Suppose

$$f = (f_n) : V = (V_n, \alpha_n) \rightarrow W = (W_n, \beta_n)$$

is a morphism. We need to check that $E(f) \in \text{Hom}_{\mathcal{P}}(E(V), E(W))$.

Consider the following diagram,

$$\begin{array}{ccccc} R_1(R_0(V_{n+1})) & \xrightarrow{s_n} & R_0(R_1(V_{n+1})) & \xrightarrow{\tilde{\alpha}_{n-1}} & R_1(V_n) \\ \downarrow R_1(R_0(f_{n+1})) & & \downarrow R_0(R_1(f_{n+1})) & & \downarrow R_1(f_n) \\ R_1(R_0(W_{n+1})) & \xrightarrow{s_n} & R_0(R_1(W_{n+1})) & \xrightarrow{\tilde{\beta}_{n-1}} & R_1(W_n) \end{array}$$

We must show that the right square commutes. The left square commutes by Lemma 4.15. The outer square commutes since f is a morphism. Since s_n is a natural isomorphism, it follows that the right square commutes. \square

4.3.2. *Adjointness of E and F.* It will be necessary for us to know that (E, F) is a bi-adjoint pair.

Theorem 4.18. The functor $F : \mathcal{M} \rightarrow \mathcal{M}$ is right adjoint to $E : \mathcal{M} \rightarrow \mathcal{M}$.

Proof. We need to check that there are natural isomorphisms:

$$\text{Hom}_{\mathcal{M}}(E(V), W) \cong \text{Hom}_{\mathcal{M}}(V, F(W))$$

for all $V, W \in \mathcal{M}$. In other words, for $n \gg 0$ we must show that there are natural isomorphism

$$(10) \quad \chi_n = \chi_n(V, W) : \text{Hom}_{G_n}(V_n, \mathbb{F}^n \otimes W_n) \rightarrow \text{Hom}_{G_n}(R_1 V_{n+1}, W_n)$$

for all $V_n, W_n \in \mathcal{M}_n$. It suffices to prove (10) for the case when V and W are homogeneous. Moreover, we can assume that $V \in \mathcal{M}(k+1)$ and $W \in \mathcal{M}(k)$ for some k , otherwise both sides of (10) are zero. Recall the induction functors I_0 and Pind_n^{n+1} (see Definitions 4.1 and 4.2). We have the following chain of isomorphisms, which holds for

$n \gg 0$:

$$\begin{aligned}
 \mathrm{Hom}_{\mathbf{G}_n}(\mathbf{R}_1 V_{n+1}, W_n) &\cong \mathrm{Hom}_{\mathbf{G}_n}(\mathbf{R}V_{n+1}, W_n) \\
 &\cong \mathrm{Hom}_{\mathbf{G}_{n+1}}(V_{n+1}, \mathrm{Pind}_n^{n+1} W_n) \\
 &\cong \mathrm{Hom}_{\mathbf{G}_{n+1}}(V_{n+1}, \mathrm{Sym}(\mathbb{F}^{n+1}) \otimes I_0(W_n)) \\
 &\cong \mathrm{Hom}_{\mathbf{G}_{n+1}}(V_{n+1}, \mathbb{F}^{n+1} \otimes I_0(W_n)) \\
 &\cong \mathrm{Hom}_{\mathbf{G}_n}(V_n, \mathbb{F}^n \otimes \mathbf{R}_0(I_0 W_n)) \\
 &\cong \mathrm{Hom}_{\mathbf{G}_n}(V_n, \mathbb{F}^n \otimes W_n).
 \end{aligned}$$

The first and fourth isomorphisms follow by degree considerations. The second since Pind_n^{n+1} is adjoint to restriction from \mathbf{G}_{n+1} to \mathbf{G}_n . The third is from Lemma 4.3. The fifth isomorphism follows since \mathbf{R}_0 defines an equivalence from $\mathcal{M}_{n+1}(\mathbf{k})$ to $\mathcal{M}_n(\mathbf{k})$ for $n \gg 0$, and from the fact that $\mathbf{R}_0(\mathbb{F}^{n+1} \otimes I_0(W_n)) \cong \mathbf{R}_0(\mathbb{F}^{n+1}) \otimes \mathbf{R}_0(I_0(W_n))$ since both are polynomial representations. Finally the last map is an isomorphism by Corollary 4.5. \square

Our next task is to show that F is also left adjoint to E . Consider V_n a rational representation of \mathbf{G}_n . We let V'_n be the contravariant dual representation of \mathbf{G}_n , i.e.,

$$(g \cdot \psi)(v) = \psi(g^t v),$$

where $\psi \in V_n^*$ and g^t is the transpose of $g \in \mathbf{G}_n$. It is easy to see V'_n is polynomial if and only if V_n is polynomial. (This is why we use g^t to define the action of \mathbf{G}_n on V_n^* , rather than the more commonly used g^{-1} .)

We define a duality functor \mathbb{D} on \mathcal{M} , by sending $V = (V_n, \alpha_n)$ to $V' = (V'_n, \alpha'_n)$, where

$$\alpha'_n = (\alpha_n^{-1})^T$$

and we are making the canonical identification $\mathbf{R}_0(V'_{n+1}) = \mathbf{R}_0(V_{n+1})'$. It is a contravariant functor, and it is easy to see that $\mathbb{D} \circ \mathbb{D} \simeq \mathrm{Id}$. Moreover, we have the following lemma.

Lemma 4.19. *There are natural isomorphisms*

$$\mathbb{D} \circ E \circ \mathbb{D} \simeq E,$$

and

$$\mathbb{D} \circ F \circ \mathbb{D} \simeq F.$$

Proof. We prove $\mathbb{D} \circ F \circ \mathbb{D} \simeq F$. The other isomorphism can be checked directly or follows from the adjunction theorem between F with E below, and the fact that \mathbb{D} is an involutive functor. For $V_n \in \mathcal{M}_n$ there is an isomorphism $(V'_n \otimes \mathbb{F}^n)' \simeq V_n \otimes \mathbb{F}^n$. Moreover, unwinding identifications one has $(\alpha'_n \otimes \iota_n)' = \alpha_n \otimes \iota_n$. Therefore $\mathbb{D} \circ F \circ \mathbb{D} \simeq F$. \square

Theorem 4.20. *The functor $F: \mathcal{M} \rightarrow \mathcal{M}$ is isomorphic to a left adjoint to $E: \mathcal{M} \rightarrow \mathcal{M}$*

Proof. We need to construct a natural isomorphism $\text{Hom}_{\mathcal{M}}(\mathbb{F}(\mathbb{V}), \mathbb{W}) \simeq \text{Hom}_{\mathcal{M}}(\mathbb{V}, \mathbb{E}(\mathbb{W}))$ for $\mathbb{V}, \mathbb{W} \in \mathcal{M}$. It is constructed by the composition of following natural isomorphisms,

$$\begin{aligned} \text{Hom}_{\mathcal{M}}(\mathbb{F}(\mathbb{V}), \mathbb{W}) &\simeq \text{Hom}_{\mathcal{M}}(\mathbb{D}(\mathbb{W}), \mathbb{D}(\mathbb{F}(\mathbb{V}))) \\ &\simeq \text{Hom}_{\mathcal{M}}(\mathbb{D}(\mathbb{W}), \mathbb{F}(\mathbb{D}(\mathbb{V}))) \\ &\simeq \text{Hom}_{\mathcal{M}}(\mathbb{E}(\mathbb{D}(\mathbb{W}), \mathbb{D}(\mathbb{V}))) \\ &\simeq \text{Hom}_{\mathcal{M}}(\mathbb{D}(\mathbb{E}(\mathbb{W})), \mathbb{D}(\mathbb{V})) \\ &\simeq \text{Hom}_{\mathcal{M}}(\mathbb{V}, \mathbb{E}(\mathbb{W})). \end{aligned}$$

Here the first and fifth isomorphisms follow from the self-duality of \mathbb{D} , the second and the fourth isomorphisms follow from Lemma 4.19, and the third isomorphism is from Theorem 4.18. \square

Fix $\mathbb{V} = (\mathbb{V}_n, \alpha_n), \mathbb{W} = (\mathbb{W}_n, \beta_n) \in \mathcal{M}$. It will be useful for us later on to have an explicit description of the isomorphism

$$\chi_n : \text{Hom}_{\mathbb{G}_n}(\mathbb{V}_n, \mathbb{F}^n \otimes \mathbb{W}_n) \rightarrow \text{Hom}_{\mathbb{G}_n}(\mathbb{R}_1(\mathbb{V}_{n+1}), \mathbb{W}_n)$$

from (10). By the proof of Theorem 4.18, χ_n is the composition $\chi_n = \tau_n \circ \kappa_n^{-1}$, where

$$\tau_n : \text{Hom}_{\mathbb{G}_{n+1}}(\mathbb{V}_{n+1}, \mathbb{F}^{n+1} \otimes I_0(\mathbb{W}_n)) \rightarrow \text{Hom}_{\mathbb{G}_n}(\mathbb{R}_1(\mathbb{V}_{n+1}), \mathbb{W}_n)$$

is the inverse map of the composition of the first four isomorphisms in the proof of Theorem 4.18 and

$$\kappa_n : \text{Hom}_{\mathbb{G}_{n+1}}(\mathbb{V}_{n+1}, \mathbb{F}^{n+1} \otimes I_0(\mathbb{W}_n)) \rightarrow \text{Hom}_{\mathbb{G}_n}(\mathbb{V}_n, \mathbb{F}^n \otimes \mathbb{W}_n)$$

is the composition of the last two isomorphisms.

We now give explicit formulas for these morphisms. By Lemma 4.3, we can view $\mathbb{F}^{n+1} \otimes I_0(\mathbb{W}_n)$ as a subspace of functions from M_{n+1} to \mathbb{W}_n . Let x_i ($i = 1, \dots, n$) be the standard basis of \mathbb{F}^{n+1} . Given a morphism $g : \mathbb{V}_{n+1} \rightarrow \mathbb{F}^{n+1} \otimes I_0(\mathbb{W}_n)$ and $v \in \mathbb{V}_{n+1}$, write $g(v) = \sum_{i=1}^{n+1} x_i \otimes g_i$, where $g_i \in I_0(\mathbb{W}_n)$. We can view g_i as functions from $M_{n,n+1}$ to \mathbb{W}_n . Then given any matrix $A = (a_{ij}) \in M_{n+1}$,

$$g(v)(A) = \left(\sum_{i=1}^{n+1} x_i \otimes g_i \right)(A) = \sum a_{n+1,i} g_i(A_{n,n+1}),$$

where $A_{n,n+1}$ is the $n \times (n+1)$ principal submatrix of A . Recall that $I_{n,n+1}$ is the $n \times (n+1)$ matrix with ones on the main diagonal and zeros elsewhere. Then we have the following lemma.

Lemma 4.21. *Given a morphism $g : \mathbb{V}_{n+1} \rightarrow \mathbb{F}^{n+1} \otimes I_0(\mathbb{W}_n)$ and $v \in \mathbb{V}_{n+1}$, write $g(v) = \sum_{i=1}^{n+1} x_i \otimes g_i$ as above.*

- (1) *If $v \in \mathbb{R}_1(\mathbb{V}_{n+1})$, then $\tau_n(g)(v) = g_{n+1}(I_{n,n+1})$.*
- (2) *If $v \in \mathbb{R}_0(\mathbb{V}_{n+1})$, then $\kappa_n(g)(\alpha_n(v)) = \sum_{i=1}^n x_i \otimes g_i(I_{n,n+1})$.*

Proof. Note that τ_n is defined via Frobenius reciprocity, and so $\tau_n(g)(v) = g(v)(I_{n+1,n+1})$. Therefore $\tau_n(g)(v) = (\sum_{i=1}^{n+1} x_i \otimes g_i)(I_{n+1,n+1})$, and the first desired formula follows. For

the second formula, by the definition of κ_n , we have

$$\begin{aligned} \kappa_n(\mathfrak{g})(\alpha_n(\mathbf{v})) &= (\pi_n \otimes \mathbf{ev}_{I_{n,n+1}})(\mathfrak{g}(\mathbf{v})) \\ &= (\pi_n \otimes \mathbf{ev}_{I_{n,n+1}})\left(\sum_{i=1}^{n+1} x_i \otimes \mathfrak{g}_i\right) \\ &= \sum_{i=1}^n x_i \otimes \mathfrak{g}_i(I_{n,n+1}). \end{aligned}$$

where π_n is the natural G_n -equivariant projection $\mathbb{F}^{n+1} \rightarrow \mathbb{F}^n$. \square

4.3.3. *An endomorphism on E.* In this section we construct an endomorphism X on E , i.e. a natural transformation $X : E \rightarrow E$. This will be used below to decompose E into sub-functors E_i .

We consider the embedding $\mathcal{U}(\mathfrak{gl}_n) \subset \mathcal{U}(\mathfrak{gl}_{n+1})$, analogous to the embedding of $G_n \subset G_{n+1}$ defined above. The Levi subgroup $G_n \times G_1$ of G_{n+1} acts on $\mathcal{U}(\mathfrak{gl}_{n+1})$, by the restriction of the adjoint action of G_{n+1} on $\mathcal{U}(\mathfrak{gl}_{n+1})$.

Set

$$X_n = \sum_{i=1}^n x_{n+1,i} x_{i,n+1} - n,$$

an element in $\mathcal{U}(\mathfrak{gl}_{n+1})$.

Lemma 4.22. *The element X_n commutes with the adjoint action of $G_n \times G_1$ on $\mathcal{U}(\mathfrak{gl}_{n+1})$, i.e.*

$$X_n \in \mathcal{U}(\mathfrak{gl}_{n+1})^{G_n \times G_1}.$$

Proof. Recall the Casimir element defined in Equation (3). We compute:

$$\begin{aligned} C_{n+1} - C_n &= \sum_{i=1}^n x_{n+1,i} x_{i,n+1} + x_{i,n+1} x_{n+1,i} + x_{n+1,n+1}^2 \\ &= 2X_n + \left(\sum_{i=1}^n x_{i,i}\right) - n x_{n+1,n+1} + x_{n+1,n+1}^2 + 2n. \end{aligned}$$

Since $C_{n+1} - C_n$ and $\left(\sum_{i=1}^n x_{i,i}\right) - n x_{n+1,n+1} + x_{n+1,n+1}^2 + 2n$ commute with $G_n \times G_1$, the result follows. \square

By the above lemma, the action of X_n on a representation $V_{n+1} \in \mathcal{M}_{n+1}$ defines an endomorphism of the restriction functors $R_i : \mathcal{M}_{n+1} \rightarrow \mathcal{M}_n$ for any $i \in \mathbb{Z}$.

Proposition 4.23. *Let $V = (V_n, \alpha_n) \in \mathcal{M}$. The maps*

$$X_n : R_1(V_{n+1}) \rightarrow R_1(V_{n+1})$$

glue to define a morphism

$$X(V) : E(V) \rightarrow E(V),$$

given by $X(V) = (X_n|_{R_1(V_{n+1})})$, so that $X : E \rightarrow E$ is a natural transformation.

Proof. Consider the following diagram,

$$\begin{array}{ccccc} R_1(R_0(V_{n+1})) & \xrightarrow{s_n} & R_0(R_1(V_{n+1})) & \xrightarrow{\tilde{\alpha}_{n-1}} & R_1(V_n) \\ \downarrow X_{n-1} & & \downarrow X_n & & \downarrow X_{n-1} \\ R_1(R_0(V_{n+1})) & \xrightarrow{s_n} & R_0(R_1(V_{n+1})) & \xrightarrow{\tilde{\alpha}_{n-1}} & R_1(V_n) \end{array}$$

We want to show that the right square commutes for $n \gg 0$. First note that the fact that the outer square commutes follows since X_{n-1} is a natural transformation from R_1 to R_1 and by definition: $\tilde{\alpha}_{n-1} = R_1(\alpha_n) \circ s_n^{-1}$. The following computation shows that the left square commutes:

$$\begin{aligned} (s_n^{-1} \circ X_n \circ s_n)|_{R_1 \circ R_0(V_{n+1})} &= \left(\sum_{i=1}^{n-1} x_{n,i} x_{i,n} + x_{n,n+1} x_{n+1,n} - n \right)|_{R_1 \circ R_0(V_{n+1})} \\ &= (X_{n-1} + x_{n,n+1} x_{n+1,n} - 1)|_{R_1 \circ R_0(V_{n+1})} \\ &= X_{n-1}|_{R_1 \circ R_0(V_{n+1})}. \end{aligned}$$

The last equality follows from Lemma A.4. Therefore the right square commutes, which shows that $X(V) \in \text{Hom}_{\mathcal{M}}(E(V), E(V))$.

Now suppose $V, W \in \mathcal{M}$ and $f \in \text{Hom}_{\mathcal{M}}(V, W)$. Since X_n acts on R_1 , it follows that $X(W) \circ E(f) = E(f) \circ X(V)$. Therefore $X : E \rightarrow E$ is a natural transformation. \square

4.3.4. *An endomorphism of F.* We now construct an explicit natural transformation Y of F , related to $X : E \rightarrow E$ by adjunction. This will be used below to decompose F into subfunctors F_i .

Fix $V, W \in \mathcal{M}$. Let $h(X)$ be the morphism induced from $X(V) : E(V) \rightarrow E(V)$ by the functor $\text{Hom}_{\mathcal{M}}(\cdot, W)$. Thus,

$$(11) \quad h(X) : \text{Hom}_{\mathcal{M}}(E(V), W) \rightarrow \text{Hom}_{\mathcal{M}}(E(V), W).$$

By adjunction (see Theorem 4.18) we obtain an endomorphism $h(X)^\vee$ of $\text{Hom}_{\mathcal{M}}(V, F(W))$ such that the following diagram commutes:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{M}}(V, F(W)) & \xrightarrow{X} & \text{Hom}_{\mathcal{M}}(E(V), W) \\ h(X)^\vee \downarrow & & \downarrow h(X) \\ \text{Hom}_{\mathcal{M}}(V, F(W)) & \xrightarrow{X} & \text{Hom}_{\mathcal{M}}(E(V), W) \end{array}$$

The map X is shorthand for $(\chi_n(V, W))_{n=0}^\infty$ (see Equation (10)). By Yoneda's Lemma we obtain a natural transformation $X^\vee : F \rightarrow F$.

To describe X^\vee more explicitly we introduce the following explicitly defined natural transformation on F . Let $Y_n \in \mathcal{U}(\mathfrak{g}_n) \otimes \mathcal{U}(\mathfrak{g}_n)$ be

$$Y_n = \sum_{1 \leq i, j \leq n} x_{i,j} \otimes x_{j,i}$$

and define $Y : F \rightarrow F$ by $Y = (Y_n)$.

Proposition 4.24. *The natural transformation $Y : F \rightarrow F$ is well-defined.*

Proof. Let $V = (V_n, \alpha_n) \in \mathcal{M}$. Recall that $F(V) = (\mathbb{F}^n \otimes V_n, \hat{\alpha}_n)$, where $\hat{\alpha}_n = \iota_n \otimes \alpha_n$. We need to show that the following diagram commutes for $n \gg 0$:

$$(12) \quad \begin{array}{ccc} \mathbf{R}_0(\mathbb{F}^{n+1} \otimes V_{n+1}) & \xrightarrow{Y_{n+1}} & \mathbf{R}_0(\mathbb{F}^{n+1} \otimes V_{n+1}) \\ \hat{\alpha}_n \downarrow & & \downarrow \hat{\alpha}_n \\ \mathbb{F}^n \otimes V_n & \xrightarrow{Y_n} & \mathbb{F}^n \otimes V_n \end{array}$$

Let $x \otimes v \in \mathbf{R}_0(\mathbb{F}^{n+1} \otimes V_{n+1})$. Then on the one hand,

$$\hat{\alpha}_n(x \otimes v) = x \otimes \alpha_n(v),$$

and therefore,

$$(13) \quad Y_n(\hat{\alpha}_n(x \otimes v)) = \sum_{1 \leq i, j \leq n} x_{i,j} \cdot x \otimes x_{j,i} \cdot \alpha_n(v).$$

On the other hand,

$$Y_{n+1}(x \otimes v) = \sum_{1 \leq i, j \leq n+1} x_{i,j} \cdot x \otimes x_{j,i} \cdot v.$$

Since $x \otimes v$ is of degree zero for the action of G_1 , this implies that $x_{i,n+1} \cdot x = 0$ for $i = 1, \dots, n+1$, and moreover that,

$$\sum_{1 \leq i \leq n+1} x_{n+1,i} \cdot x \otimes x_{i,n+1} \cdot v = 0.$$

Therefore,

$$(14) \quad \hat{\alpha}_n \circ Y_{n+1}(x \otimes v) = \sum_{1 \leq i, j \leq n} x_{i,j} \cdot x \otimes \alpha_n(x_{j,i} \cdot v).$$

Since α_n is a G_n morphism, it commutes with $x_{j,i}$ for $1 \leq i, j \leq n$, and so (13) agrees with (14), and diagram (12) commutes.

It is trivial to show that Y is compatible with morphisms $f : V \rightarrow W$ in \mathcal{M} . \square

Fix $V, W \in \mathcal{M}$. Let

$$(15) \quad h(Y)^\circ : \mathrm{Hom}_{\mathcal{M}}(V, F(W)) \rightarrow \mathrm{Hom}_{\mathcal{M}}(V, F(W))$$

be the map induced by applying the functor $\mathrm{Hom}_{\mathcal{P}}(V, \cdot)$ to the morphism $Y(W) : F(W) \rightarrow F(W)$.

Theorem 4.25. *In $\mathrm{End}(F)$, $X^\vee = Y$.*

Proof. Let $V, W \in \mathcal{M}$. By the definition of X^\vee , to show the equality $X^\vee = Y$ it suffices to show that the following diagram commutes:

$$(16) \quad \begin{array}{ccc} \mathrm{Hom}_{\mathcal{M}}(V, F(W)) & \xrightarrow{X} & \mathrm{Hom}_{\mathcal{M}}(E(V), W) \\ h(Y)^\circ \downarrow & & \downarrow h(X) \\ \mathrm{Hom}_{\mathcal{M}}(V, F(W)) & \xrightarrow{X} & \mathrm{Hom}_{\mathcal{M}}(E(V), W) \end{array}$$

To show that diagram (16) commutes we need to show that for $n \gg 0$ the following diagram commutes,

$$\begin{array}{ccc} \mathrm{Hom}_{G_n}(V_n, \mathbb{F}^n \otimes W_n) & \xrightarrow{\chi_n} & \mathrm{Hom}_{G_n}(R_1 V_{n+1}, W_n) \\ \downarrow h(Y_n)^\circ & & \downarrow h(X_n) \\ \mathrm{Hom}_{G_n}(V_n, \mathbb{F}^n \otimes W_n) & \xrightarrow{\chi_n} & \mathrm{Hom}_{G_n}(R_1 V_{n+1}, W_n) \end{array}$$

Since $\chi_n = \tau_n \circ \kappa_n^{-1}$, to check that the above diagram commutes we check that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Hom}_{G_n}(V_n, \mathbb{F}^n \otimes W_n) & \xleftarrow{\kappa_n} & \mathrm{Hom}_{G_{n+1}}(V_{n+1}, \mathbb{F}^{n+1} \otimes I_0(W_n)) & \xrightarrow{\tau_n} & \mathrm{Hom}_{G_n}(R_1 V_{n+1}, W_n) \\ \downarrow h(Y_n)^\circ & & \downarrow h(Y_{n+1})^\circ & & \downarrow h(X_n) \\ \mathrm{Hom}_{G_n}(V_n, \mathbb{F}^n \otimes W_n) & \xleftarrow{\kappa_n} & \mathrm{Hom}_{G_{n+1}}(V_{n+1}, \mathbb{F}^{n+1} \otimes I_0(W_n)) & \xrightarrow{\tau_n} & \mathrm{Hom}_{G_n}(R_1 V_{n+1}, W_n) \end{array}$$

By Lemma 4.21, we have explicit formulas for κ_n and τ_n , which we will now use.

First we check the right diagram commutes, i.e., we need to check $\tau_n(Y_{n+1} \circ g) = \tau_n(g) \circ X_n$, for any morphism $g : V_{n+1} \rightarrow \mathbb{F}^{n+1} \otimes I_0(W_n)$.

Let $v \in R_1(V_{n+1})$ and set $g(v) = \sum_{i=1}^{n+1} x_i \otimes g_i$. Then we have

$$\begin{aligned} (Y_{n+1} \circ g)(v) &= Y_{n+1}(g(v)) \\ &= \sum_{k,l=1}^{n+1} \sum_{i=1}^{n+1} x_{k,l} \cdot x_i \otimes x_{l,k} \cdot g_i \\ &= \sum_{i,k=1}^{n+1} x_k \otimes x_{i,k} \cdot g_i. \end{aligned}$$

By Lemma 4.21, $\tau_n(Y_{n+1} \circ g)(v) = \sum_{i=1}^{n+1} (x_{i,n+1} \cdot g_i)(I_{n,n+1})$. Then by Equation (30) of Lemma A.6, $\tau_n(Y_{n+1} \circ g)(v) = \sum_{i=1}^n (x_{i,n+1} \cdot g_i)(I_{n,n+1})$.

Now we will compute $(\tau_n(g) \circ X_n)(v) = \tau_n(g)(X_n(v))$. For this, first we look at $g(X_n(v))$. Since X_n commutes with G_n on $R_1(V_{n+1})$, then

$$\begin{aligned} g(X_n(v)) &= X_n(g(v)) \\ &= \sum_{k=1}^n x_{n+1,k} (x_{k,n+1} (\sum_{i=1}^{n+1} x_i \otimes g_i)) - n g(v) \\ &= n x_{n+1} \otimes g_{n+1} + \sum_{k=1}^n (x_k \otimes x_{n+1,k} \cdot g_{n+1} + x_{n+1} \otimes x_{k,n+1} \cdot g_k \\ &\quad + \sum_{i=1}^{n+1} x_i \otimes x_{n+1,k} \cdot (x_{k,n+1} \cdot g_i)) - n g(v) \end{aligned}$$

Applying Lemma 4.21 again, we have

$$\tau_n(g)(X_n(v)) = \sum_{k=1}^n (x_{k,n+1} \cdot g_k)(I_{n,n+1}) + (x_{n+1,k} \cdot (x_{k,n+1} \cdot g_{n+1}))(I_{n,n+1}).$$

By Equation (31) of Lemma A.6, we have $\tau_n(\mathfrak{g})(X_n(\mathfrak{v})) = \sum_{k=1}^n (x_{k,n+1} \cdot \mathfrak{g}_k)(I_{n,n+1})$. Hence the equality $\tau_n(Y_{n+1} \circ \mathfrak{g}) = \tau_n(\mathfrak{g}) \circ X_n$ holds. This shows that the right diagram commutes.

Finally we check that the left diagram also commutes. We have to check that

$$(\mathfrak{h}(Y_n) \circ \kappa_n)(\mathfrak{g}) = (\kappa_n \circ \mathfrak{h}(Y_{n+1}))(\mathfrak{g}).$$

Since Y_{n+1} gives a natural transformation on the tensor functor $\mathbb{F}^{n+1} \otimes \cdot$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{F}^{n+1} \otimes W_{n+1} & \xrightarrow{Y_{n+1}} & \mathbb{F}^{n+1} \otimes W_{n+1} \\ \downarrow 1 \otimes \beta_n^\vee & & \downarrow 1 \otimes \beta_n^\vee \\ \mathbb{F}^{n+1} \otimes I_0(W_n) & \xrightarrow{Y_{n+1}} & \mathbb{F}^{n+1} \otimes I_0(W_n) \end{array}$$

where β_n^\vee is the morphism induced from $\beta_n : R_0(W_{n+1}) \rightarrow W_n$. Combined with the commutativity of diagram (12) and Lemma 4.7, the commutativity of left diagram follows. \square

4.3.5. *The functors E_α and F_α .* Finally we are ready to define the family of functors $\{E_\alpha, F_\alpha\}$. For a vector space \mathbf{U} over \mathbb{F} , an operator $T : \mathbf{U} \rightarrow \mathbf{U}$ and a scalar $\alpha \in \mathbb{F}$, we denote by $\mathbf{U}[\alpha]$ the α -generalized eigenspace of T on \mathbf{U} , i.e.

$$\mathbf{U}[\alpha] = \bigcup_{N>0} \ker(T - \alpha)^N.$$

For a morphism $f : V \rightarrow V$, where $V \in \mathcal{M}$, we can also define the notion of a generalized eigenspace. Indeed, if $f = (f_n)$ and $V = (V_n, \alpha_n)$, then we set $V[\alpha] = (V_n[\alpha], \alpha_n)$, where $V_n[\alpha]$ is the α -generalized eigenspace of f_n on V_n . It is straightforward to check that the morphisms α_n restrict to give gluing data $R_0(V_{n+1}[\alpha]) \rightarrow V_n[\alpha]$.

Lemma 4.26. *Let $V \in \mathcal{M}$, $f \in \text{End}_{\mathcal{M}}(V)$, and $\alpha \in \mathbb{F}$. Then there exists $N > 0$ such that*

$$V[\alpha] = \ker(f - \alpha)^N.$$

Proof. Any V in \mathcal{M} admits a composition series of finite length, and by Lemma 4.11, f acts on each subquotient by some scalar. Hence the lemma follows. \square

Definition 4.27. *For $V \in \mathcal{M}$ and $\alpha \in \mathbb{F}$ set*

$$E_\alpha(V) = E(V)[\alpha],$$

the α -generalized eigenspace of $X(V) : E(V) \rightarrow E(V)$. This defines a functor

$$E_\alpha : \mathcal{M} \rightarrow \mathcal{M}.$$

It follows by Proposition 4.23 that the functors E_α are well-defined, and we now have a decomposition E into sub-functors:

$$E = \bigoplus_{\alpha \in \mathbb{F}} E_\alpha.$$

We define sub-functors F_α of F completely analogously.

Definition 4.28. For $V \in \mathcal{M}$ and $\mathfrak{a} \in \mathbb{F}$ set

$$F_{\mathfrak{a}}(V) = F(V)[\mathfrak{a}],$$

the \mathfrak{a} -generalized eigenspace of $Y : F(V) \rightarrow F(V)$. This defines a functor

$$F_{\mathfrak{a}} : \mathcal{M} \rightarrow \mathcal{M}.$$

It follows by Proposition 4.24 that the functors $F_{\mathfrak{a}}$ are well-defined, and we now have a decomposition F into sub-functors:

$$F = \bigoplus_{\mathfrak{a} \in \mathbb{F}} F_{\mathfrak{a}}.$$

Proposition 4.29. For every \mathfrak{a} , $(E_{\mathfrak{a}}, F_{\mathfrak{a}})$ is an adjoint pairs of functors.

Proof. Given objects V and W in the category \mathcal{M} , by the definition of $E_{\mathfrak{a}}$ and $F_{\mathfrak{a}}$ and Lemma 4.26, there exists a positive integer N such that,

$$E_{\mathfrak{a}}(V) = \ker(X - \mathfrak{a})^N : E(V) \rightarrow E(V),$$

$$F_{\mathfrak{a}}(W) = \ker(Y - \mathfrak{a})^N : F(W) \rightarrow F(W).$$

Recall that we defined morphisms h and h^0 in (11) and (15). Consider the following diagram:

$$(17) \quad \begin{array}{ccc} \mathrm{Hom}_{\mathcal{M}}(V, F_{\mathfrak{a}}(W)) & \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(E_{\mathfrak{a}}(V), W) \\ \mathrm{h}(\ker(Y - \mathfrak{a})^N)^{\circ} \downarrow & & \downarrow \mathrm{h}(\ker(X - \mathfrak{a})^N) \\ \mathrm{Hom}_{\mathcal{M}}(V, F(W)) & \xrightarrow{\chi} & \mathrm{Hom}_{\mathcal{M}}(E(V), W) \\ \mathrm{h}((Y - \mathfrak{a})^N)^{\circ} \downarrow & & \downarrow \mathrm{h}((X - \mathfrak{a})^N) \\ \mathrm{Hom}_{\mathcal{M}}(V, F(W)) & \xrightarrow{\chi} & \mathrm{Hom}_{\mathcal{M}}(E(V), W) \end{array}$$

To be precise, for example $\mathrm{h}(\ker(X - \mathfrak{a})^N)$ is obtained by applying $\mathrm{Hom}_{\mathcal{M}}(\cdot, W)$ to the morphism $\ker(X - \mathfrak{a})^N : E(V) \rightarrow E(V)$, and then restricting the resulting morphism to $\mathrm{Hom}_{\mathcal{M}}(E_{\mathfrak{a}}(V), W)$. In the above diagram, by Theorem 4.18, χ is an isomorphism. By Theorem 4.25, the bottom square commutes. Since $E_{\mathfrak{a}}(V)$ is a direct summand of $E(V)$, $\mathrm{h}(\ker(X - \mathfrak{a})^N)$ is the kernel of $\mathrm{h}((X - \mathfrak{a})^N)$. So the isomorphism χ induces an isomorphism from $\mathrm{Hom}(V, F_{\mathfrak{a}}(W))$ to $\mathrm{Hom}(E_{\mathfrak{a}}(V), W)$. \square

5. CATEGORIFYING THE FOCK SPACE

In this section we prove our main theorem. First, in Section 5.1, we recall the definitions of the degenerate affine Hecke algebra and Chuang and Rouquier's notion of categorification. In Section 5.2 we define the data of a \mathfrak{g} -categorification on \mathcal{M} and prove our main result, namely that this data is indeed a \mathfrak{g} -categorification which categorifies the Fock space representation of \mathfrak{g} . By Chuang-Rouquier theory we deduce derived equivalences between blocks of \mathcal{M} . In Section 5.3 we recover the crystal of Fock space from the set of simple objects of \mathcal{M} .

5.1. The definition of \mathfrak{g} -categorification. We recall Chuang and Rouquier's notion of \mathfrak{g} -categorification following [Ro].

Definition 5.1. *Let \bar{H}_n be the degenerate affine Hecke algebra of GL_n . As an abelian group*

$$\bar{H}_n = \mathbb{Z}[y_1, \dots, y_n] \otimes \mathbb{Z}\mathfrak{S}_n.$$

The algebra structure is defined as follows: $\mathbb{Z}[y_1, \dots, y_n]$ and $\mathbb{Z}\mathfrak{S}_n$ are subalgebras, and the following relations hold between the generators of these subalgebras:

$$\tau_i y_j = y_j \tau_i \text{ if } |i - j| \geq 1$$

and

$$(18) \quad \tau_i y_{i+1} - y_i \tau_i = 1$$

(here $\tau_1, \dots, \tau_{n-1}$ are the simple transpositions of $\mathbb{Z}\mathfrak{S}_n$).

For an abelian \mathbb{F} -linear category \mathcal{V} , let $K(\mathcal{V})$ denote the *complexified* Grothendieck group of \mathcal{V} . The equivalence class of an object $V \in \mathcal{V}$ is denoted $[V] \in K(\mathcal{V})$, and given an exact functor $F : \mathcal{V} \rightarrow \mathcal{V}$, $[F] : K(\mathcal{V}) \rightarrow K(\mathcal{V})$ denotes the induced linear operator.

Definition 5.2 (Definition 5.29, [Ro]). *Let \mathcal{V} be an abelian \mathbb{F} -linear category. A \mathfrak{g} -categorification on \mathcal{V} is the data of:*

- (1) an adjoint pair (E, F) of exact functors $\mathcal{V} \rightarrow \mathcal{V}$,
- (2) $X \in \text{End}(E)$ and $T \in \text{End}(E^2)$,
- (3) a decomposition $\mathcal{V} = \bigoplus_{\omega \in \mathfrak{P}} \mathcal{V}_\omega$.

Let $X^\vee \in \text{End}(F)$ be the endomorphism of F induced by adjunction. Then given $i \in \mathbb{F}$ let E_i (resp. F_i) be the generalized i -eigensubfunctor of X (resp. X^\vee) acting on E (resp. F). We assume that

- (4) $E = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} E_i$,
- (5) The action of $\{[E_i], [F_i]\}_{i \in \mathbb{Z}/p\mathbb{Z}}$ on $K(\mathcal{V})$ gives rise to an integrable representation of \mathfrak{g} ,
- (6) For all i , $E_i(\mathcal{V}_\omega) \subset \mathcal{V}_{\omega + \alpha_i}$ and $F_i(\mathcal{V}_\omega) \subset \mathcal{V}_{\omega - \alpha_i}$,
- (7) F is isomorphic to the left adjoint of E ,
- (8) The degenerate affine Hecke algebra \bar{H}_n acts on $\text{End}(E^n)$ via

$$(19) \quad y_i \mapsto E^{n-i} X E^{i-1} \text{ for } 1 \leq i \leq n,$$

and

$$(20) \quad \tau_i \mapsto E^{n-i-1} T E^{i-1} \text{ for } 1 \leq i \leq n-1.$$

Remark 5.3. *To clarify notation, the natural endomorphism y_i of E^n assigns to $V \in \mathcal{V}$ an endomorphism of $E^n(V)$ as follows: apply the functor E^{n-i} to the morphism*

$$X(E^{i-1}(V)) : E^i(V) \rightarrow E^i(V).$$

Remark 5.4. *Rouquier defines a 2-Kac Moody algebra $\mathfrak{A}(\mathfrak{g})$ associated to \mathfrak{g} , and shows that a \mathfrak{g} -categorification \mathcal{V} is equivalent to a 2-representation of $\mathfrak{A}(\mathfrak{g})$, i.e. a 2-functor $\mathcal{V} : \mathfrak{A}(\mathfrak{g}) \rightarrow \text{Cat}$ (cf. Theorem 5.30 in [Ro]).*

5.2. The \mathfrak{g} -categorification on \mathcal{M} . By now we've defined functors E and F on \mathcal{M} (cf. Section 4.3), and shown that these are bi-adjoint (Theorems 4.18 and 4.20). We also have a natural endomorphism $X \in \text{End}(E)$ (Proposition 4.23), and we've shown that $X^\vee = Y$ (cf. Theorem 4.25). Now we introduce the remaining data necessary to define a \mathfrak{g} -categorification, and finally prove our main theorem.

5.2.1. *The \mathfrak{g} -action on $K(\mathcal{M})$.* The vector space $K(\mathcal{M})$ has a basis $[V(\lambda)]$, where $\lambda \in \Lambda$. Therefore we have a natural linear isomorphism

$$\xi : K(\mathcal{M}) \rightarrow \mathcal{F}$$

given by $\xi([V(\lambda)]) = v_\lambda$, where \mathcal{F} is the Fock space representation of \mathfrak{g} (cf. Section 2.2). In this section we show that ξ intertwines the operators $[E_i]$ and $[F_i]$ acting on $K(\mathcal{M})$ with the Chevalley generators e_i and f_i acting on \mathcal{F} . Consequently the operators $[E_i]$ and $[F_i]$ induce an action of \mathfrak{g} on $K(\mathcal{M})$, and ξ is an isomorphism of \mathfrak{g} -modules.

Lemma 5.5. *Suppose $V_k \in \mathcal{M}_k(\mathfrak{k})$ has a Weyl filtration*

$$0 = V_k^0 \subset V_k^1 \subset \cdots \subset V_k^N = V_k,$$

where for $i = 0, \dots, N-1$

$$V_k^{i+1}/V_k^i \simeq V_k(\mu_i).$$

Then $V = \Psi^{-1}(V_k) \in \mathcal{M}(\mathfrak{k})$ has a Weyl filtration

$$0 = V^0 \subset V^1 \subset \cdots \subset V^N = V$$

where for $i = 0, \dots, N-1$.

$$V^{i+1}/V^i \simeq V(\mu_i).$$

Proof. Note that $V_k^i \in \mathcal{M}_k(\mathfrak{k})$ for $i = 0, \dots, N$. Set

$$V^i = \Psi^{-1}(V_k^i).$$

By definition $V^i \in \mathcal{M}(\mathfrak{k})$, and we have a filtration

$$0 = V^0 \subset V^1 \subset \cdots \subset V^N = V.$$

Now $\Psi(V^{i+1}/V^i) \simeq V_k^{i+1}/V_k^i \simeq V_k(\mu_i)$. Therefore by Propositions 4.9 and 4.13,

$$V^{i+1}/V^i \simeq V(\mu_i).$$

□

Proposition 5.6. *Let $\lambda \in \Lambda$ and set $V = V(\lambda)$. Then,*

- (1) *The object $E(V)$ admits a Weyl filtration*

$$0 = E(V)^0 \subset E(V)^1 \subset \cdots \subset E(V)^N = E(V).$$

The composition factors that occur in this filtration are isomorphic to $V(\mu)$ for all $\mu \in \Lambda$ such that $\mu \longrightarrow \lambda$, and each such factor occurs exactly once.

- (2) *The object $F(V)$ admits a Weyl filtration*

$$0 = F(V)^0 \subset F(V)^1 \subset \cdots \subset F(V)^N = F(V).$$

The composition factors that occur in this filtration are isomorphic to $V(\mu)$ for all $\mu \in \Lambda$ such that $\lambda \longrightarrow \mu$, and each such factor occurs exactly once.

Proof. Let \mathfrak{k} be such that $\lambda \in \Lambda_{\mathfrak{k}}$. We have the following diagram, which commutes:

$$\begin{array}{ccc} \mathcal{M}(\mathfrak{k}) & \xrightarrow{\Psi_{\mathfrak{k}}} & \mathcal{M}_{\mathfrak{k}}(\mathfrak{k}) \\ \downarrow E & & \downarrow R_1 \\ \mathcal{M}(\mathfrak{k}-1) & \xrightarrow{\Psi_{\mathfrak{k}-1}} & \mathcal{M}_{\mathfrak{k}-1}(\mathfrak{k}-1) \end{array}$$

Note that by Proposition 3.7 the functors Ψ_k and Ψ_{k-1} are equivalences. Write $V = (V_n) \in \mathcal{M}(k)$. Then, by this commutative square and Proposition 4.9,

$$V \simeq \Psi^{-1}(R_1(V_k)).$$

By the above lemma and Lemma A.3, part (1) of the proposition follows. The proof of part (2) of the proposition is entirely analogous with the above square replaced by

$$\begin{array}{ccc} \mathcal{M}(k) & \xrightarrow{\Psi_{k+1}} & \mathcal{M}_{k+1}(k) \\ F \downarrow & & \downarrow \mathbb{F}^{k+1} \otimes \\ \mathcal{M}(k+1) & \xrightarrow{\Psi_{k+1}} & \mathcal{M}_{k+1}(k+1) \end{array}$$

□

Lemma 5.7. *Let $\lambda \in \Lambda$ and $V = V(\lambda) \in \mathcal{M}$, and consider the Weyl filtrations of $E(V)$ and $F(V)$ as in Proposition 5.6. Then X (resp. Y) preserves the filtration of $E(V)$ (resp. $F(V)$). Moreover,*

- (1) *Given $0 \leq i \leq N-1$ set $\mu \in \Lambda, j \in \mathbb{Z}/p\mathbb{Z}$ such that $E(V)^{i+1}/E(V)^i \simeq V(\mu)$ and $\mu \xrightarrow{j} \lambda$. Then X acts on $E(V)^{i+1}/E(V)^i$ by j .*
- (2) *Given $0 \leq i \leq N-1$ set $\mu \in \Lambda, j \in \mathbb{Z}/p\mathbb{Z}$ such that $F(V)^{i+1}/F(V)^i \simeq V(\mu)$ and $\lambda \xrightarrow{j} \mu$. Then Y acts on $F(V)^{i+1}/F(V)^i$ by j .*

In particular, $E_j = F_j = 0$ for all $j \in \mathbb{F}$ such that $j \notin \mathbb{Z}/p\mathbb{Z}$, i.e.

$$E = \bigoplus_{j \in \mathbb{Z}/p\mathbb{Z}} E_j$$

and

$$F = \bigoplus_{j \in \mathbb{Z}/p\mathbb{Z}} F_j.$$

Proof. It is clear from the formula

$$(21) \quad C_{n+1} - C_n = 2X_n + \left(\sum_{i=1}^n x_{i,i} \right) - nx_{n+1,n+1} + x_{n+1,n+1}^2 + 2n$$

that X preserves the filtration of $E(V)$. Now let $V = (V_n)$. Set k so that $\lambda \in \Lambda_k$ and let $n \geq k$. Consider first a Weyl filtration of $R_1(V_{n+1})$:

$$0 = R_1(V_{n+1})^0 \subset R_1(V_{n+1})^1 \subset \cdots \subset R_1(V_{n+1})^N = R_1(V_{n+1})$$

such that

$$R_1(V_{n+1})^{i+1}/R_1(V_{n+1})^i \simeq V_n(\mu).$$

(cf. Lemma A.3). Since $\mu \xrightarrow{j} \lambda$ there exists ℓ such that $\lambda_\ell - \ell = j$. By (4) one computes

$$c_{n+1}(\lambda) - c_n(\mu) = 2(\lambda_\ell - \ell) + k + n.$$

Since the degree of representation V_{n+1} is k , which is the size of the partition λ , then $\sum_{i=1}^{n+1} x_{i,i}$ acts by k on V_{n+1} . On the other hand, by the definition of R_1 , x_{n+1} acts on $R_1(V_{n+1})$ by 1. Then by (21) it follows that X_n acts on $R_1(V_{n+1})^{i+1}/R_1(V_{n+1})^i \simeq V_n(\mu)$ by

$$(c_{n+1}(\lambda) - c_n(\mu) - k - n)/2 = \lambda_\ell - \ell = j.$$

Hence X acts on $E(V)^{i+1}/E(V)^i$ by j . This proves the statement (1) of the lemma.

The statement (2) of the lemma follows along similar lines. Firstly, it is clear that Y preserves the filtration of $F(V)$. Now considering Y_n as an element of $U(\mathfrak{g}_n) \otimes U(\mathfrak{g}_n)$, note that

$$(22) \quad Y_n = \frac{1}{2}(\Delta(C_n) - C_n \otimes 1 - 1 \otimes C_n)$$

where $\Delta : U(\mathfrak{g}_n) \rightarrow U(\mathfrak{g}_n) \otimes U(\mathfrak{g}_n)$ is the coproduct in $U(\mathfrak{g}_n)$. Consider a Weyl filtration of $F(V)_n = \mathbb{F}^n \otimes V_n$:

$$0 = F(V)_n^0 \subset F(V)_n^1 \subset \cdots \subset F(V)_n^N = F(V)_n$$

such that

$$F(V)_n^{i+1}/F(V)_n^i \simeq V_n(\mu).$$

By (22) and Lemmas A.3 and 2.4, it follows that Y_n acts on $F(V)_n^{i+1}/F(V)_n^i$ by j .

The last statement of the lemma follows from parts (1) and (2) and the fact that Weyl objects descend to a basis of the Grothendieck group $K(\mathcal{M})$. \square

Recall that $\xi : K(\mathcal{M}) \rightarrow \mathcal{F}$ is defined by $\xi([V(\lambda)]) = v_\lambda$. As an immediate corollary of the above lemma we obtain:

Proposition 5.8. *For every $i \in \mathbb{Z}/\mathfrak{p}\mathbb{Z}$ the following diagram commutes:*

$$\begin{array}{ccc} K(\mathcal{M}) & \xrightarrow{\xi} & \mathcal{F} \\ [E_i] \downarrow & & \downarrow e_i \\ K(\mathcal{M}) & \xrightarrow{\xi} & \mathcal{M} \end{array}$$

Similarly, there is a commutative square with $[E_i]$ and e_i replaced by $[F_i]$ and f_i , respectively.

In particular, the operators $[E_i]$ and $[F_i]$ define an action of \mathfrak{g} on $K(\mathcal{M})$ via $e_i \mapsto [E_i]$ and $f_i \mapsto [F_i]$, and ξ is an isomorphism of \mathfrak{g} -modules.

5.2.2. *The \bar{H}_n -action on $\text{End}(E^n)$.* To define the action of the degenerate affine Hecke algebra on powers of E we first need to define the operator T on E^2 . Set $t_n = s_{n+1}$. Clearly for $V_{n+2} \in \mathcal{M}_{n+2}$, t_n defines an operator on $R_1^2(V_{n+2})$. Let $T = (t_n)_{n=0}^\infty$.

Lemma 5.9. *The operator T acts on $E \circ E$, i.e. $T \in \text{End}(E^2)$.*

Proof. Given $V = (V_n, \alpha_n)$, for $n \gg 0$, we need to check the following diagram commutes:

$$\begin{array}{ccc} R_0(E^2(V)_n) & \xrightarrow{\tilde{\alpha}_{n-1}} & E^2(V)_{n-1} \\ \downarrow t_n & & \downarrow t_{n-1} \\ R_0(E^2(V)_n) & \xrightarrow{\tilde{\alpha}_{n-1}} & E^2(V)_{n-1} \end{array}$$

Thus it suffices to check that the following diagram commutes:

$$\begin{array}{ccc} R_0 \circ R_1 \circ R_1(V_{n+1}) & \xrightarrow{\alpha_{n+1} \circ s_{n+1}^{-1} \circ s_n^{-1}} & R_1 \circ R_1(V_{n+1}) \\ \downarrow s_{n+1} & & \downarrow s_n \\ R_0 \circ R_1 \circ R_1(V_{n+2}) & \xrightarrow{\alpha_{n+1} \circ s_{n+1}^{-1} \circ s_n^{-1}} & R_1 \circ R_1(V_{n+1}) \end{array}$$

This diagram commutes by the braid relation, $s_n s_{n+1} s_n = s_{n+1} s_n s_{n+1}$, and the fact that α_{n+1} commutes with s_n (α_{n+1} is a morphism of \mathbf{G}_{n+1} -modules). \square

Proposition 5.10. *The degenerate affine Hecke algebra \bar{H}_n acts on $\text{End}(\mathbb{E}^n)$ via formulas (19) and (20).*

Proof. First note that for $V_{n+2} \in \mathcal{M}_{n+2}$, t_n^2 acts on $\mathbb{R}_1^2(V_{n+2})$ by the identity. Therefore $T^2 = 1$ in $\text{End}(\mathbb{E}^2)$. The only other relation that is not trivial to show is relation (18). Relation (18) is a consequence of the following equality in $\text{End}(\mathbb{E}^2)$:

$$T \circ XE - EX \circ T = 1.$$

This equality follows from the identity

$$(23) \quad s_n X_{n-1} - X_n s_n = 1$$

in $\text{End}(\mathbb{R}_1^2)$. Since we have,

$$s_n X_n s_n^{-1} = X_{n-1} + x_{n,n+1} x_{n+1,n} - 1$$

equation (23) follows from Lemma A.5. \square

5.2.3. *The decomposition of \mathcal{M} as a direct sum of weight categories.* Let $\mathcal{F} = \bigoplus_{\omega \in \mathbf{P}} \mathcal{F}_\omega$ be the weight space decomposition as a representation of \mathfrak{g} . Recall that for a partition $\lambda \in \Lambda$, v_λ is a weight vector. We define a *weight function* $\text{wt} : \Lambda \rightarrow \mathbf{P}$ by requiring that $v_\lambda \in \mathcal{F}_{\text{wt}(\lambda)}$.

We recall some combinatorial notions. For a nonnegative integer d , let Λ_d denote the set of partitions of d . A partition λ is a *p-core* if there exist no $\mu \subset \lambda$ such that the skew-partition λ/μ is a rim p -hook. By definition, if $p = 0$ then all partitions are p -cores. Given a partition λ , we denote by $\tilde{\lambda}$ the p -core obtained by successively removing all rim p -hooks. We define the number $(|\lambda| - |\tilde{\lambda}|)/p$ to be the p -weight of λ .

Define an equivalence relation \sim on Λ_d by decreeing $\lambda \sim \mu$ if $\tilde{\lambda} = \tilde{\mu}$.

Let $\lambda, \mu \in \Lambda_d$. As a consequence of (11.6) in [KL] we have

$$(24) \quad \tilde{\lambda} = \tilde{\mu} \iff \text{wt}(\lambda) = \text{wt}(\mu).$$

Therefore we index the set of equivalence classes Λ_d / \sim by weights in \mathbf{P} , i.e. a weight $\omega \in \mathbf{P}$ corresponds to a subset (possibly empty) of Λ_d .

Let $\text{Irr}\mathcal{M}_d$ denote the set of simple objects in \mathcal{M}_d up to isomorphism. This set is naturally identified with Λ_d . We say two simple objects in \mathcal{M}_d are *adjacent* if they occur as composition factors of some indecomposable object in \mathcal{M}_d . Consider the equivalence relation \approx on $\text{Irr}\mathcal{M}_d$ generated by adjacency. Via the identification of $\text{Irr}\mathcal{M}_d$ with Λ_d we obtain an equivalence relation \approx on Λ_d .

Theorem 5.11 (Theorem 2.12, [D]). *The equivalence relations \sim and \approx on Λ_d are the same.*

By the above theorem, Equation (24) and Equation (6) we can label any block of \mathcal{M} by weights $\omega \in \mathbf{P}$, and the p -weight of a block is well-defined. So we have the decomposition,

$$(25) \quad \mathcal{M} = \bigoplus_{\omega} \mathcal{M}_\omega.$$

5.2.4. *The \mathfrak{g} -categorification on \mathcal{M} .* We can now state and prove our main result:

Theorem 5.12. *The data of*

- (1) *the adjoint pair of functors (E, F) ,*
- (2) *$X \in \text{End}(E)$ and $T \in \text{End}(E^2)$, and*
- (3) *the decomposition of $\mathcal{M} = \bigoplus_{\omega \in \mathbb{P}} \mathcal{M}_\omega$*

is a \mathfrak{g} -categorification on the abelian \mathbb{F} -linear category \mathcal{M} .

Proof. The adjointness of (E, F) is Theorem 4.18. The endomorphism X on E is defined in Proposition 4.23, while T is defined in Lemma 5.9. The decomposition of \mathcal{M} into subcategories is Equation (25).

Now we must check that conditions (4)-(8) of Definition 5.2 are satisfied. Condition (4) follows from Lemma 5.7. Now, since $X^\vee = Y$ (cf. Theorem 4.25), the functors F_i we defined (cf. Definition 4.28) agree with the functors F_i that arise as generalized eigenspaces of X^\vee acting on F . Therefore, conditions (5) and (6) are a consequence of Proposition 5.8. Condition (7) is Theorem 4.20, while condition (8) is Proposition 5.10. \square

By [K, Section 12], any weight ω appearing in Fock space is of the form $\sigma(\omega_0) - \ell\delta$, where ω_0 is the first fundamental weight and σ is some element in the affine Weyl group of \mathfrak{g} . By [K1, Proposition, 11.1.5], ℓ is exactly the p -weight of the corresponding block. Therefore the weights of any two blocks are conjugate by some element of the affine Weyl group if and only if they have the same p -weight. As a consequence of Chuang-Rouquier theory we obtain:

Corollary 5.13. *If two blocks of \mathcal{M} have the same p -weight then they are derived equivalence.*

5.3. **The Misra-Miwa crystal from the category \mathcal{M} .** We now show how to recover the Misra-Miwa crystal of Fock space (cf. [MM]) from the category \mathcal{M} . We first briefly recall the definition of this crystal. For this we need to first also recall several combinatorial notions (see [BK2] for a more thorough discussion of these notions).

Label all the i -addable boxes of a partition λ by $+$ and all i -removable boxes by $-$. The i -signature of λ is the sequence of $+$ and $-$ obtained by going along the rim of the Young diagram from bottom left to top right and reading off all the signs. The reduced i -signature of λ is obtained from the i -signature by successively erasing all neighboring pairs of the form $+-$. Note the reduced i -signature always looks like a sequence of $-$'s followed by $+$'s. Boxes corresponding to a $-$ in the reduced i -signature are called i -normal, boxes corresponding to a $+$ are called i -conormal. The rightmost i -normal box (corresponding to the rightmost $-$ in the reduced i -signature) is called i -good, and the leftmost i -conormal box (corresponding to the leftmost $+$ in the reduced i -signature) is called i -cogood.

The Misra-Miwa crystal of Fock space (cf. [LLT]) consists of the set Λ together with maps

$$\text{wt} : \Lambda \rightarrow \mathbb{P}, \tilde{e}_i, \tilde{f}_i : \Lambda \rightarrow \Lambda \cup \{0\}, \epsilon_i, \phi_i : \Lambda \rightarrow \mathbb{Z}$$

defined as follows:

- (1) The wt function is defined above in Section 5.2.3.
- (2) The operator \tilde{e}_i is given by the rule: if there exists an i -good box for λ , then $\tilde{e}_i(\lambda) = \mu$, where μ is obtained by removing this i -good box from λ ; otherwise $\tilde{e}_i(\lambda) = 0$.

- (3) The operator \tilde{f}_i is given by the rule: if there exists an i -cogood box for λ , then $\tilde{f}_i(\lambda) = \mu$ where μ is obtained from λ by adding this i -cogood box; otherwise $\tilde{f}_i(\lambda) = 0$.
- (4) $\epsilon_i(\lambda)$ is the number of i -normal boxes of λ
- (5) $\phi_i(\lambda)$ is the number of i -conormal boxes of λ .

We now reformulate a theorem of Brundan and Kleshchev [BK2] in our setting.

Theorem 5.14. *For any simple object $L(\lambda)$ in \mathcal{M} , if λ has an i -cogood box, then the socle of $F_i(L(\lambda))$ is $L(\mu)$, where μ is obtained from λ by adding the i -cogood box. Otherwise $F_i(L(\lambda)) = 0$.*

Proof. The functor $F : \mathcal{M} \rightarrow \mathcal{M}$ is given by $F = (F_n)$, where $F_n : \mathcal{M}_n \rightarrow \mathcal{M}_n$ is tensoring with the standard module \mathbb{F}^n . Similarly, $F_i = ((F_i)_n)$, where $(F_i)_n$ is the generalized i -eigen-subfunctor of Y_n .

It suffices to show that given a partition λ , then for $n \gg 0$ the socle of $(F_i)_n(L_n(\lambda))$ is $L_n(\lambda + \epsilon_k)$ if the box $(k, \lambda_k + 1)$ is cogood and $\lambda_k + 1 - k = i$ (here ϵ_k denotes the weight $(0, \dots, 1, \dots, 0)$, where the one is in the k^{th} position), and otherwise it is 0. In [BK2], the translation functor Tr^i is introduced, and by Theorem A(i) [BK2], it is easy to see that Tr^i coincides with $(F_i)_n$. Then Theorem B(i) in [BK2] implies our theorem. \square

Corollary 5.15. *For any simple object $L(\lambda)$ in \mathcal{M} , if λ has an i -good box, then the socle of $E_i(L(\lambda))$ is $L(\mu)$, where μ is obtained from λ by deleting the i -good box. Otherwise $E_i(L(\lambda)) = 0$.*

Proof. Consider two simple objects $L(\lambda)$ and $L(\mu)$. By Proposition 4.29, we have

$$\text{Hom}_{\mathcal{M}}(E_i(L(\lambda)), L(\mu)) = \text{Hom}_{\mathcal{M}}(L(\lambda), F_i(L(\mu))).$$

Recall that on the category \mathcal{M} , there is a contravariant duality \mathbb{D} , which maps any simple object to itself. By Lemma 4.19, \mathbb{D} commutes with E (and hence with E_i), so therefore

$$\text{Hom}_{\mathcal{M}}(E_i(L(\lambda)), L(\mu)) = \text{Hom}_{\mathcal{M}}(L(\mu), E_i(L(\lambda)))$$

and so we have

$$(26) \quad \text{Hom}_{\mathcal{M}}(L(\lambda), F_i(L(\mu))) = \text{Hom}_{\mathcal{M}}(L(\mu), E_i(L(\lambda))).$$

Now note that μ is obtained from λ by deleting the i -good box if and only if λ is obtained from μ by adding this box (which is i -cogood for μ). Therefore, by the above theorem, Schur's lemma (Lemma 4.11), and equation (26), the corollary follows. \square

Let \mathbb{B} be the set of isomorphism classes of simple objects in \mathcal{M} , namely

$$\mathbb{B} = \{L(\lambda) : \lambda \in \Lambda\}.$$

By the above theorem and corollary, we can define operators

$$\tilde{E}_i, \tilde{F}_i : \mathbb{B} \rightarrow \mathbb{B} \cup \{0\}$$

as follows: for any simple object $L(\lambda)$, set $\tilde{E}_i(L(\lambda))$ be the socle of $E_i(L(\lambda))$, and set $\tilde{F}_i(L(\lambda))$ be the socle of $F_i(L(\lambda))$. Moreover, let

$$\epsilon_i(L(\lambda)) = \max\{m : \tilde{E}_i^m(L(\lambda)) \neq 0\}$$

and

$$\phi_i(L(\lambda)) = \max\{m : \tilde{F}_i^m(L(\lambda)) \neq 0\}$$

Finally, set $\text{wt}(L(\lambda)) = \text{wt}(\lambda)$. By the above theorem and corollary, we obtain:

Theorem 5.16. *The data $(\mathbb{B}, \tilde{\mathbb{E}}_i, \tilde{\mathbb{F}}_i, \varepsilon_i, \varphi_i, \mathbf{wt})$ defines the crystal of Fock space, which is isomorphic to the Misra-Miwa crystal $(\Lambda, \tilde{\mathbf{e}}_i, \tilde{\mathbf{f}}_i, \varepsilon_i, \phi_i, \mathbf{wt})$ described above.*

APPENDIX A. THE GENERAL LINEAR GROUP

In this appendix we collect some standard/technical results that we use in the body of the paper. In Section A.1 we recall some standard combinatorics related to Weyl modules and their interaction with certain tensor and restriction functors. In Section A.2 we prove some technical lemmas.

A.1. Weyl filtrations.

Theorem A.1. (1) *Let V_{n+1} be a Weyl module. Then the G_n -module $R_i(V_{n+1})$ admits a Weyl filtration for all $i \in \mathbb{Z}$ (Proposition II.4.24, [J]).*
 (2) *For any $V_n, W_n \in \text{Rep}(G_n)$ admitting Weyl filtrations, the G_n -module $V_n \otimes W_n$ also admits a Weyl filtration (Proposition II.4.21, [J]).*

We will make use of the following classical result about branching from G_{n+1} to G_n in characteristic zero. (Recall that if $\mathfrak{p} = 0$ then $\text{Rep}(G_{n+1})$ is a semisimple category.)

Lemma A.2. *Suppose $\mathfrak{p} = 0$ and let $V \in \text{Rep}(G_{n+1})$ be a simple module.*

(1) *The G_n -module $R(V)$ is a multiplicity-free. In particular, if $\mathfrak{n} \geq \mathfrak{k}$ and $\lambda \in \Lambda_{\mathfrak{k}}$ then*

$$(27) \quad R_1(V_{n+1}(\lambda)) \simeq \bigoplus V_n(\mu)$$

the sum over all $\mu \in \Lambda_{\mathfrak{k}-1}$ such that $\mu \longrightarrow \lambda$ (cf. Theorem 8.1.1, [GW]).

(2) *The G_n -module $\mathbb{F}^n \otimes V$ is a multiplicity-free. Precisely,*

$$(28) \quad \mathbb{F}^n \otimes V_n(\lambda) \simeq \bigoplus V_n(\mu)$$

where the sum is over all μ such that $\lambda \longrightarrow \mu$ (cf. Corollary 9.2.4, [GW]).

Now we prove the analogue of this lemma for positive characteristic.

Lemma A.3. *Let $\mathfrak{n} \geq \mathfrak{k}$, $\lambda \in \Lambda_{\mathfrak{k}}$ and consider the Weyl module $V_n(\lambda)$.*

(1) *Then $R_1(V_{n+1}(\lambda))$ has a Weyl filtration and the corresponding Weyl factors occur with multiplicity one and have precisely those highest weights that appear in the right side of (27).*
 (2) *Similarly, $\mathbb{F}^n \otimes V_n(\lambda)$ has a Weyl filtration and the corresponding Weyl factors occur with multiplicity one and have precisely those highest weights that appear in the right side of (28).*

Proof. The modules under consideration have Weyl filtrations by Theorem A.1. Therefore, as elements of the integral group algebra $\mathbb{Z}[X(T_n)]$, the characters $\text{char}_{\mathbb{F}}(R_1(V_{n+1}(\lambda)))$ and $\text{char}_{\mathbb{F}}(V_n(\lambda) \otimes \mathbb{F}^n)$ do not depend on the characteristic of \mathbb{F} . In characteristic zero we know by the above lemma that

$$\text{char}_{\mathbb{C}}(R_1(V_{n+1}(\lambda))) = \sum \text{char}_{\mathbb{C}}(V_n(\mu))$$

the sum over all μ such that $\mu \longrightarrow \lambda$, and

$$\text{char}_{\mathbb{C}}(V_n(\lambda) \otimes \mathbb{C}^n) = \sum \text{char}_{\mathbb{C}}(V_n(\mu))$$

the sum over all μ such that $\lambda \longrightarrow \mu$. Therefore, the same formulas hold with \mathbb{C} replaced by \mathbb{F} , and the lemma follows. \square

A.2. Some technical lemmas.

Lemma A.4. *Let $V_{n+1} \in \text{Rep}(G_{n+1})$. Then the operator $x_{n,n+1} \cdot x_{n+1,n}$ acts on the space $R_1 \circ R_0(V_{n+1})$ by 1.*

Proof. Let $i, j \in \mathbb{Z}$. First we show that $x_{n,n+1}$ defines an operator:

$$(29) \quad x_{n,n+1} : R_i R_j(V_{n+1}) \rightarrow R_{i+1} \circ R_{j-1}(V_{n+1}).$$

Indeed, for $k \in \{1, \dots, n+1\}$, let $\zeta_k : \mathbb{F}^\times \rightarrow G_{n+1}$ be the one-parameter subgroup

$$z \mapsto \text{diag}(1, \dots, z, \dots, 1),$$

where z occurs in the k^{th} position. Now suppose $v \in R_i \circ R_j(V_{n+1})$ and $z \in \mathbb{F}^\times$. Then,

$$\begin{aligned} \zeta_n(z) \cdot x_{n,n+1} \cdot v &= \zeta_n(z) \cdot x_{n,n+1} \cdot \zeta_n(z^{-1}) \cdot \zeta_n(z) \cdot v \\ &= z^{i+1} x_{n,n+1} \cdot v \end{aligned}$$

and similarly $\zeta_{n+1}(z) \cdot x_{n,n+1} \cdot v = z^{j-1} x_{n+1,n} \cdot v$. This proves (29).

By (29), $x_{n,n+1} : R_1 \circ R_0(V_{n+1}) \rightarrow R_2 \circ R_{-1}(V_{n+1})$. Since V_{n+1} is a polynomial representation, $R_2 \circ R_{-1}(V_{n+1}) = 0$, and therefore $x_{n,n+1} \cdot v = 0$ for all $v \in R_1 \circ R_0(V_{n+1})$. Therefore for $v \in R_1 \circ R_0(V_{n+1})$,

$$x_{n,n+1} x_{n+1,n} \cdot v = (x_{n,n} - x_{n+1,n+1}) \cdot v = v.$$

\square

Lemma A.5. *Let V_{n+1} be a representation of G_{n+1} . Then the following identity of operators holds on $R_1 \circ R_1(V_{n+1})$:*

$$1 - x_{n+1,n} x_{n,n+1} = s_n^{-1}.$$

Proof. By similar methods as applied in the proof of the above lemma, it follows that $x_{n,n+1}^2 \cdot v = 0$ for $v \in R_1 \circ R_1(V_{n+1})$. Moreover, such v are of weight $(1, 1)$ relative to the torus of GL_2 . By the representation theory of GL_2 , if $\text{char}(\mathbb{F}) > 2$ then, all polynomial representation of GL_2 of degree 2 is semisimple. Hence

$$R_1 \circ R_1(V_{n+1}) \subset I^{(1,1)} \oplus I^{(2,0)}$$

where $I^{(i,j)}$ is the isotypic component of $V_{n+1}|_{G_{n-1}}$ corresponding to the irreducible representation of GL_2 of highest weight (i, j) .

For any $v \in R_1 \circ R_1(V_{n+1})$, we decompose v as $v = v_{1,1} + v_{2,0}$, where $v_{(i,j)} \in I^{(i,j)}$. In particular $v_{(2,0)}$ lies in the $(1, 1)$ -weight space of I^2 . By elementary theory of \mathfrak{gl}_2 , it is easy to check that for either $v_{(1,1)}$ or $v_{(2,0)}$,

$$(1 - x_{n+1,n} x_{n,n+1})(v_{(i,j)}) = s_n^{-1}(v_{(i,j)}).$$

\square

Lemma A.6. *Let $g : V_{n+1} \rightarrow \mathbb{F}^{n+1} \otimes I_0(W_n)$ be a morphism of G_{n+1} -modules. If $v \in R_1(V_{n+1})$ and set $g(v) = \sum_{i=1}^{n+1} x_i \otimes g_i$, then*

$$(30) \quad x_{n+1,n+1} \cdot g_i = \begin{cases} g_i & \text{if } i = 1, \dots, n \\ 0 & \text{if } i = n+1 \end{cases}$$

$$(31) \quad x_{k,n+1} \cdot g_{n+1} = 0 \text{ if } k = 1, \dots, n$$

Proof. By hypothesis, $x_{n+1,n+1} \cdot v = v$, and since g is a morphism of G_{n+1} -modules, $x_{n+1,n+1} \cdot g(v) = f(v)$. Therefore,

$$\sum_{i=1}^n x_i \otimes x_{n+1,n+1} \cdot g_i + x_{n+1} \otimes g_{n+1} + x_{n+1} \otimes x_{n+1,n+1} \cdot g_{n+1} = \sum_{i=1}^{n+1} x_i \otimes g_i,$$

which implies formula (30).

To prove the second formula, recall that $G_1 \subset G_{n+1}$ acts on $I_0(W_n)$ semi-simply. For an element $t \in G_1$, we have

$$t \cdot (x_{k,n+1} \cdot g_{n+1}) = tx_{k,n+1}t^{-1} \cdot t \cdot g_{n+1} = t^{-1}(x_{k,n+1} \cdot g_{n+1}),$$

and so $x_{k,n+1} \cdot g_{n+1}$ is of weight -1 for the action of G_1 . But $I_0(W_n)$ is a polynomial G_{n+1} -representation, so in particular all the weights of G_1 on $I_0(W_n)$ are nonnegative. Therefore $x_{k,n+1} \cdot g_{n+1}$ has to be zero, for $k = 1, \dots, n$. \square

APPENDIX B. POLYNOMIAL FUNCTORS

B.1. The functor category. Recall that \mathbb{F} is algebraically closed. The category of finite dimensional vector spaces over \mathbb{F} is denoted $\mathbf{Vect}_{\mathbb{F}}$. In [FS], Friedlander and Suslin introduce the category of *strict polynomial functors of finite degree*. Their category, whose objects consists of certain endofunctors of $\mathbf{Vect}_{\mathbb{F}}$, will be denoted by \mathcal{P} .

For $V, W \in \mathbf{Vect}_{\mathbb{F}}$, *polynomial maps* from V to W are by definition elements of $S(V^*) \otimes W$, where $S(V^*)$ denotes the symmetric algebra of the linear dual of V . Elements of $S^d(V^*) \otimes W$ are said to be of *degree d*.

Definition B.1. *The objects of the category \mathcal{P} are functors $T : \mathbf{Vect}_{\mathbb{F}} \rightarrow \mathbf{Vect}_{\mathbb{F}}$ that satisfy the following properties:*

- (1) *for any $V, W \in \mathbf{Vect}_{\mathbb{F}}$, the map of vector spaces*

$$\mathrm{Hom}_{\mathbb{F}}(V, W) \rightarrow \mathrm{Hom}_{\mathbb{F}}(T(V), T(W))$$

is polynomial, and

- (2) *the degree of the map*

$$\mathrm{End}_{\mathbb{F}}(V) \rightarrow \mathrm{End}_{\mathbb{F}}(T(V))$$

is bounded uniformly for all $V \in \mathbf{Vect}_{\mathbb{F}}$.

The morphisms in \mathcal{P} are natural transformations of functors.

B.2. The canonical equivalence. We now show that the categories \mathcal{M} and \mathcal{P} are canonically equivalent.

Let $T \in \mathcal{P}$. By functoriality $T(\mathbb{F}^n)$ carries an algebraic action of G_n . The representation $T(\mathbb{F}^n)$ is polynomial (Proposition 3.8, [F]). There exists a canonical functor $\Phi : \mathcal{P} \rightarrow \mathcal{M}$ defined as follows:

$$\Phi(T) = (T(\mathbb{F}^n), \alpha_n)_{n=0}^{\infty},$$

where $\alpha_n : R_0(T(\mathbb{F}^{n+1})) \rightarrow T(\mathbb{F}^n)$ is the map induced from the natural G_n -equivariant projection $\pi_n : \mathbb{F}^{n+1} \rightarrow \mathbb{F}^n$,

$$\begin{array}{ccc} T(\mathbb{F}^{n+1}) & \xrightarrow{T(\pi_n)} & T(\mathbb{F}^n) \\ \uparrow & \nearrow \alpha_n & \\ R_0(T(\mathbb{F}^{n+1})) & & \end{array}$$

We need to show that α_n is an isomorphism for any n , and thus $(T(\mathbb{F}^n), \alpha_n)$ is a well-defined object in \mathcal{M} .

Let $\Gamma^k \in \mathcal{M}$ be the k -th divided power of vector spaces, i.e. $\Gamma^k(\mathbf{V}) = (\otimes^k \mathbf{V})^{S_k}$, and let $\Gamma^{k,n}$ be the polynomial functor $\Gamma^k \circ \text{Hom}_{\mathbb{F}}(\mathbb{F}^n, \cdot)$. Note that the action of G_n on \mathbb{F}^n induces an action on $\text{Hom}_{\mathbb{F}}(\mathbb{F}^n, \cdot)$ and hence on $\Gamma^{k,n}$.

Lemma B.2. *The map $\alpha_n : R_0(T(\mathbb{F}^{n+1})) \rightarrow T(\mathbb{F}^n)$ is an isomorphism, and thus the assignment $T \mapsto (T(\mathbb{F}^n), \alpha_n)_{n=0}^{\infty}$ defines a functor $\Phi : \mathcal{P} \rightarrow \mathcal{M}$.*

Proof. We can assume T is of degree k . The G_n action on the functor $\Gamma^{k,n}$ induces a representation of G_n on the vector space $\text{Hom}_{\mathcal{P}}(\Gamma^{k,n}, T)$. By Theorem 2.10 in [FS], $T(\mathbb{F}^n)$ is canonically isomorphic to $\text{Hom}_{\mathcal{P}}(\Gamma^{k,n}, T)$ as G_n -modules.

Thus we need to check that $R_0(\text{Hom}_{\mathcal{P}}(\Gamma^{k,n}, T)) \simeq \text{Hom}_{\mathcal{P}}(\Gamma^{k,n-1}, T)$. The functor $\Gamma^{k,n}$ can be decomposed canonically as

$$\Gamma^{k,n} = \bigoplus_{k_1+k_2+\dots+k_n=k} \Gamma^{k_1} \otimes \dots \otimes \Gamma^{k_n}.$$

By Corollary 2.12 in [FS], $\Gamma^{k_1} \otimes \dots \otimes \Gamma^{k_n}$ exactly represents the weight space of $T(\mathbb{F}^n)$ with weight (k_1, k_2, \dots, k_n) . In other words,

$$\text{Hom}_{\mathcal{P}}(\Gamma^{k_1} \otimes \dots \otimes \Gamma^{k_n}, T) \simeq T(\mathbb{F}^n)^{k_1, \dots, k_n}$$

where $T(\mathbb{F}^n)^{k_1, \dots, k_n}$ is the weight space corresponding to the character (k_1, \dots, k_n) . Hence

$$\begin{aligned} R_0(\text{Hom}_{\mathcal{P}}(\Gamma^{k,n}, T)) &\simeq \bigoplus_{k_1+k_2+\dots+k_{n-1}=k} \text{Hom}_{\mathcal{P}}(\Gamma^{k_1} \otimes \dots \otimes \Gamma^{k_{n-1}}, T) \\ &\simeq \text{Hom}_{\mathcal{P}}(\Gamma^{k,n-1}, T). \end{aligned}$$

□

Proposition B.3. *The functor $\Phi : \mathcal{P} \rightarrow \mathcal{M}$ is an equivalence.*

Proof. Let $\mathcal{P}(k)$ be the category of strict polynomial functors of degree k . By Lemma 2.6 in [FS], $\mathcal{P} = \bigoplus_{d=0}^{\infty} \mathcal{P}(k)$. Recall that $\mathcal{M} = \bigoplus_{n=0}^{\infty} \mathcal{M}(k)$. It is clear that Φ preserves the degree, i.e. $\Phi(\mathcal{P}(k)) \subset \mathcal{M}(k)$.

Let $\Phi_n(k) : \mathcal{P}(k) \rightarrow \mathcal{M}_n(k)$ be the functor mapping T to $T(\mathbb{F}^n)$. Obviously $\Phi_n(k) = \Psi_n(k) \circ \Phi$, i.e. the following diagram commutes,

$$\begin{array}{ccc} \mathcal{P}(k) & \xrightarrow{\Phi} & \mathcal{M}(k) \\ \Phi_n(k) \downarrow & \swarrow \Psi_n(k) & \\ \mathcal{M}_n(k) & & \end{array}$$

To prove $\Phi : \mathcal{P} \rightarrow \mathcal{M}$ is an equivalence, it is enough to prove that $\Phi : \mathcal{P}(k) \rightarrow \mathcal{M}(k)$ is an equivalence for every $k \geq 0$. To show this, for each k , simply choose some $n \geq k$. Then by Lemma 3.4 in [FS] and Proposition 3.7, $\Phi_n(k)$ and $\Psi_n(k)$ are both equivalences. It follows that $\Phi : \mathcal{P}(k) \rightarrow \mathcal{M}(k)$ is also. □

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