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Tensor invariants, saturation problems, and Dynkin automorphisms

Jiuzu Hong [a,*], Linhui Shen [b]

[a] *Department of Mathematics, Yale University, New Haven, CT 06511, United States*
[b] *Department of Mathematics, Northwestern University, Evanston, IL 60208, United States*

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ABSTRACT

Let G be a connected almost simple algebraic group with a Dynkin automorphism σ. Let G_σ be the connected almost simple algebraic group associated with G and σ. We prove that the dimension of the tensor invariant space of G_σ is equal to the trace of σ on the corresponding tensor invariant space of G. We prove that if G has the saturation property then so does G_σ. As a consequence, we show that the spin group $\mathrm{Spin}(2n+1)$ has saturation factor 2, which strengthens the results of Belkale–Kumar [1] and Sam [28] in the case of type B_n.

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* Corresponding author.
E-mail addresses: hjzzjh@gmail.com (J. Hong), linhui.shen@northwestern.edu (L. Shen).

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1. Introduction

Let G be a connected almost simple algebraic group with a Dynkin automorphism σ. One can associate with it another almost simple algebraic group G_σ (see Section 2.2). We investigate the relation between the tensor invariant spaces of G and G_σ in this paper.

In fact we can identify the dominant weights of G_σ and the σ-invariant dominant weights of G. Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be a sequence of dominant weights of G_σ. Denote by V_{λ_i} (respectively W_{λ_i}) the irreducible representation of G (respectively G_σ) of highest weight λ_i. We are interested in the pair of tensor invariant spaces

$$V_{\underline{\lambda}}^G := (V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n})^G, \quad W_{\underline{\lambda}}^{G_\sigma} := (W_{\lambda_1} \otimes \ldots \otimes W_{\lambda_n})^{G_\sigma}. \tag{1}$$

1.1. Main results

We present two main results relating $V_{\underline{\lambda}}^G$ and $W_{\underline{\lambda}}^{G_\sigma}$.

1.1.1. Twining formula

Let λ be a dominant weight of G_σ. The Dynkin automorphism σ uniquely determines an action σ on the representation V_λ of G by keeping the highest weight vectors invariant. Let μ be a weight of G_σ. The twining character formula asserts that the trace of σ on the weight space $V_\lambda(\mu)$ is equal to the dimension of $W_\lambda(\mu)$.

The twining character formula is originally due to Jantzen [11]. Since then, there has been many different proofs appearing in the literature (e.g. [6,26,27,20,10]). One of these approaches uses natural bases of the representations that are compatible with the action σ. It was achieved via canonical basis in [20], and via MV cycles in [10]. Due to the works of Lusztig [22], Berenstein–Zelevinsky [2] and Kamnitzer [13], canonical basis and MV cycles can be parameterized by many different but equivalent combinatorial objects, i.e., Lusztig's data, BZ patterns, and MV polytopes. These parameterizations are crucially used in the proofs of [20] and [10].

The first result of the present paper provides an analogue of the twining formula in the setting of tensor invariant spaces. Note that the Dynkin automorphism σ determines an action σ on $V_{\underline{\lambda}}^G$. Our first main theorem is as follows.

Theorem 1.1. *The trace of σ on the space $V_{\underline{\lambda}}^G$ is equal to the dimension of $W_{\underline{\lambda}}^{G_\sigma}$:*

$$\text{trace } (\sigma : V_{\underline{\lambda}}^G \to V_{\underline{\lambda}}^G) = \dim W_{\underline{\lambda}}^{G_\sigma}. \tag{2}$$

Theorem 1.1 is proved in Section 5.1. We remark here that Theorem 1.1 implies similar twining formulas for more general multiplicity spaces.

1.1.2. Saturation property

We say that a reductive group G has saturation factor k if

- for any dominant weights $\lambda_1, \lambda_2, \cdots, \lambda_m$ such that $\sum_{i=1}^m \lambda_i$ is in the root lattice of G, if $(V_{N\lambda_1} \otimes V_{N\lambda_2} \otimes \cdots \otimes V_{N\lambda_m})^G \neq 0$ for some positive integer N, then $(V_{k\lambda_1} \otimes V_{k\lambda_2} \otimes \cdots \otimes V_{k\lambda_n})^G \neq 0$.

Kapovich–Millson [16] proved that every almost simple group has saturation property but with a wild factor. There is a general saturation conjecture asserting that every simply-laced group has saturation factor 1 [15]. When $G = \mathrm{SL}_n$, it was first proved by Knutson–Tao [17] using honeycombs. A different proof was due to Derksen–Weyman [3]. When $G = \mathrm{Spin}(8)$, it was proved by Kapovich–Kumar–Millson [14]. It is still open for simply-laced groups of other types. For a more thorough survey on saturation problems, see [19, Section 8].

The second result of this paper shows that the saturation property of G implies the saturation property of G_σ.

Theorem 1.2. *If G has saturation factor k, then G_σ has saturation factor $c_\sigma k$, where*

$$c_\sigma = \begin{cases} 2 & \text{if } G \text{ is not of type } A_{2n} \text{ and } \sigma \text{ is of order } 2, \\ 3 & \text{if } \sigma \text{ is of order } 3, \\ 4 & \text{if } \sigma \text{ is of order } 2 \text{ and } G \text{ is of type } A_{2n}. \end{cases} \tag{3}$$

Theorem 1.2 is proved in Section 5.2.

Non-simply-laced groups are expected to have saturation factor 2. For such groups not of type G_2, if we assume the saturation conjecture of simply-laced groups, then it follows from Theorem 1.2. In particular, the works of Knutson–Tao and Derksen–Weyman imply that

Corollary 1.3. *The spin group* $\mathrm{Spin}(2n+1)$ *has saturation factor* 2.

Proof. By Example 2.1 in Section 2.2, if $G = \mathrm{SL}_{2n}$ and σ is nontrivial, then $G_\sigma = \mathrm{Spin}(2n+1)$. Theorem 1.2 implies that $\mathrm{Spin}(2n+1)$ has saturation factor 2. □

The idea that the saturation property of a big group implies the saturation property of the intimately related small group was also adopted by Belkale–Kumar [1], in which they showed that Knutson–Tao's theorem implies that the saturation factors of $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(2n)$ are 2. However, the techniques used by them are very different from ours.

1.2. Main methods

The main methods of this paper are the geometric Satake correspondence [21,7,25] and the work of Goncharov–Shen [8] on parameterizations of bases of tensor invariant spaces.

Let $G^\vee$ be the Langlands dual group of G. Let $\mathcal{K} := \mathbb{C}((t))$ and let $\mathcal{O} := \mathbb{C}[[t]]$. We consider the *affine Grassmannian* of the Langlands dual group

$$\mathrm{Gr}_{G^\vee} := G^\vee(\mathcal{K})/G^\vee(\mathcal{O}).$$

The geometric Satake correspondence provides a connection between the geometry of the affine Grassmannian of $G^\vee$ and the representation theory of G. As a consequence, the top components of certain *cyclic convolution variety* of $G^\vee$ provides a basis of the corresponding tensor invariant space of G (Lemma 4.6). Following Fontaine–Kamnitzer–Kuperberg [5], we call it the *Satake basis* of G.

Another main tool is certain *tropical points* introduced by Goncharov–Shen [8]. The tropical points are obtained via the *tropicalization* of the *configuration space of decorated flags* of G. We call these tropical points *G-laminations*, whose definition is recalled in Sections 3.2–3.5. One of the main results of Goncharov–Shen is that there exists a canonical bijection between G-laminations and the Satake basis of $G^\vee$ (Theorem 4.2). When $G = \mathrm{PGL}_2$, the G-laminations are exactly the integral laminations on a polygon [4, Section 12]. When $G = \mathrm{GL}_n$, the G-laminations encapsulate the *hives* [8, Section 3]. In this sense the result of Goncharov–Shen generalizes the work of Kamnitzer [12] that hives parameterize the Satake bases of tensor invariant spaces of GL_n.

We emphasize that Theorem 1.2 is proved by essentially using combinatorial properties of G-laminations. The geometric Satake correspondence is only used to establish that the cardinality of the set of $G^\vee$-laminations equals the tensor product multiplicity.

1.3. Strategies

Let $\underline{\lambda}$ be a tuple of dominant weights of G_σ. Denote by $\mathcal{B}_{\underline{\lambda},G}$ the Satake basis of $V_{\underline{\lambda}}^G$. Denote by $\mathbf{C}_{\underline{\lambda},G^\vee}$ the set of the $G^\vee$-laminations that parameterize $\mathcal{B}_{\underline{\lambda},G}$. Denote by $\mathcal{B}_{\underline{\lambda},G_\sigma}$ and $\mathbf{C}_{\underline{\lambda},(G_\sigma)^\vee}$ the corresponding sets for the small group G_σ.

The Satake basis $\mathcal{B}_{\underline{\lambda},G}$ has remarkable properties. One of them is that, there exists a Dynkin automorphism σ of G such that the induced action on $V_{\underline{\lambda}}^G$ interchanges the elements in the Satake basis $\mathcal{B}_{\underline{\lambda},G}$ (Proposition 4.8). Thus the trace of σ on $V_{\underline{\lambda}}^G$ equals the number of σ-invariant elements in $\mathcal{B}_{\underline{\lambda},G}$.

The Dynkin automorphism σ of G gives rise to a Dynkin automorphism $\sigma^\vee$ of $G^\vee$. The latter induces an automorphism $\sigma^\vee$ on the set of $G^\vee$-laminations (see Section 3.4). Theorem 4.4 and Proposition 4.8 assert that the $\sigma^\vee$-action on $G^\vee$-laminations is compatible with the σ-action on the Satake basis of G, i.e., the following diagram commutes:

$$\begin{tikzcd} \mathbf{C}_{\underline{\lambda},G^\vee} \arrow[r, "\simeq"] \arrow[d, "\sigma^\vee"] & \mathcal{B}_{\underline{\lambda},G} \arrow[d, "\sigma"] \\ \mathbf{C}_{\underline{\lambda},G^\vee} \arrow[r, "\simeq"] & \mathcal{B}_{\underline{\lambda},G} \end{tikzcd} \tag{4}$$

Therefore the $\sigma^\vee$-invariant $G^\vee$-laminations are in bijection with the σ-invariant elements in the Satake basis of G.

Theorem 3.25 is one of the main technical results for proving Theorem 1.1. It asserts that the $\sigma^\vee$-invariant $G^\vee$-laminations are in one-to-one correspondence with the $(G_\sigma)^\vee$-laminations, i.e., there exists a canonical bijection

$$(\mathbf{C}_{\underline{\lambda},G^\vee})^{\sigma^\vee} \simeq \mathbf{C}_{\underline{\lambda},(G_\sigma)^\vee}. \tag{5}$$

Combining (4) and (5), the σ-invariant elements in $\mathcal{B}_{\underline{\lambda},G}$ are in bijection with the elements in $\mathcal{B}_{\underline{\lambda},G_\sigma}$. Theorem 1.1 follows as a direct consequence.

Theorem 3.29 provides a summation map

$$\Sigma:\ \mathbf{C}_{\underline{\lambda},G^\vee} \longrightarrow (\mathbf{C}_{c_\sigma\cdot\underline{\lambda},G^\vee})^{\sigma^\vee} \simeq \mathbf{C}_{c_\sigma\cdot\underline{\lambda},(G_\sigma)^\vee}, \tag{6}$$

where c_σ is the number appearing in Theorem 1.2. If the set $\mathbf{C}_{\underline{\lambda},G^\vee}$ is nonempty, then the set $\mathbf{C}_{c_\sigma\cdot\underline{\lambda},(G_\sigma)^\vee}$ is also nonempty. In this way we prove Theorem 1.2 (see Section 5.2).

1.4. Other applications

Along our proofs of Theorems 1.1, 1.2, we get several interesting numerical results of representation theory related to G and G_σ.

Proposition 1.4. *With the same setting as in Theorem 1.1 and Theorem 1.2, we have*

1. $\dim V^G_{\underline{\lambda}} \geq \dim W^{G_\sigma}_{\underline{\lambda}}$.
2. *If* $\dim V^G_{\underline{\lambda}} = 1$, *then* $\dim W^{G_\sigma}_{\underline{\lambda}} = 1$.
3. *If* $\dim V^G_{\underline{\lambda}} \neq 0$, *then* $\dim W^{G_\sigma}_{c_\sigma \cdot \underline{\lambda}} \neq 0$.

Proof. The first and the second results follow from Theorem 3.25. The third result follows from Theorem 3.29. □

Recall a conjecture by W. Fulton asserting that for a triple $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ of dominant weights of GL_n, if $\dim V^{\mathrm{GL}_n}_{\underline{\lambda}} = 1$, then $\dim V^{\mathrm{GL}_n}_{N\underline{\lambda}} = 1$ for all $N \in \mathbb{N}$. The conjecture was proved by Knutson–Tao–Woodward [18] using honeycomb models. Combining it with Proposition 1.4 (2), we get the following result.

Proposition 1.5. *Let* $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ *be a triple of* σ*-invariant dominant weights of* SL_n. *If* $\dim V^{\mathrm{SL}_n}_{\underline{\lambda}} = 1$, *then for all* $N \in \mathbb{N}$ *we have*

$$\begin{cases} \dim(W_{N\underline{\lambda}})^{\mathrm{Spin}(n+1)} = 1 & \textit{if } n \textit{ is even} \\ \dim(W_{N\underline{\lambda}})^{\mathrm{Sp}(n-1)} = 1 & \textit{if } n \textit{ is odd} \end{cases}, \tag{7}$$

where $\mathrm{Sp}(n-1)$ *is the symplectic group.*

2. Basics of reductive groups

Let G be a connected almost simple group with a Dynkin automorphism σ. In this section, we introduce two different groups G_σ and G^σ related to G.

2.1. Dynkin automorphisms of G

Let G be a connected almost simple algebraic group over $\mathbb{C}$. Let T be a maximal torus in G and let B be a Borel subgroup containing T. Denote by $X^\vee$ and X the lattices of cocharacters and characters of T. We associate a root datum $(X^\vee, X, \alpha_i^\vee, \alpha_i, i \in I)$ with (G, B, T) together with a perfect pairing

$$\langle\ ,\ \rangle : X^\vee \times X \to \mathbb{Z}.$$

Here I is the index set of simple coroots $\{\alpha_i^\vee\}$ and simple roots $\{\alpha_i\}$. We have the Cartan matrix $(a_{ij}) := \big(\langle \alpha_i^\vee, \alpha_j \rangle\big)$.

A diagram automorphism σ of the root datum $(X^\vee, X, \alpha_i^\vee, \alpha_i, i \in I)$ consists of automorphisms of $X^\vee$ and of X, and a permutation of I (without confusion, all of them are denoted by σ) such that

1. $\langle \sigma(\lambda^\vee), \sigma(\mu) \rangle = \langle \lambda^\vee, \mu \rangle$ for any $\lambda^\vee \in X^\vee$ and $\mu \in X$.
2. $\sigma(\alpha_i) = \alpha_{\sigma(i)}$ and $\sigma(\alpha_i^\vee) = \alpha^\vee_{\sigma(i)}$.

Let $x_i : \mathbb{C} \to G$ and $y_i : \mathbb{C} \to G$ be root subgroups associated with the simple roots α_i and $-\alpha_i$. The datum $(T, B, x_i, y_i; i \in I)$ is called a pinning of G if it gives rise to a homomorphism $\gamma_i : \mathrm{SL}_2 \to G$ for each $i \in I$ such that

$$\gamma_i(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}) = x_i(a), \quad \gamma_i(\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}) = y_i(a), \quad \gamma_i(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}) = \alpha_i^\vee(a). \tag{8}$$

Let σ be an automorphism of G that preserves B and T. It induces a diagram automorphism of the root datum $(X^\vee, X, \alpha_i, \alpha_i^\vee; i \in I)$, which is still denoted by σ. We call σ a *Dynkin automorphism* of G if it preserves a pinning of G, i.e.,

$$\sigma\big(x_i(a)\big) = x_{\sigma(i)}(a), \quad \sigma\big(y_i(a)\big) = y_{\sigma(i)}(a), \quad \sigma\big(\alpha_i^\vee(a)\big) = \alpha_{\sigma(i)}^\vee(a), \qquad \forall i \in I.$$

By the isomorphism theorem of the theory of reductive groups (e.g. [29, Section 9]), every diagram automorphism arises from a Dynkin automorphism of G.

2.2. The associated group G_σ

Every diagram automorphism σ of a root datum $(X^\vee, X, \alpha_i^\vee, \alpha_i, i \in I)$ gives rise to the following datum:

1. Let X_σ be the lattice of σ-fixed elements in X. Let $X_\sigma^\vee := \mathrm{Hom}_{\mathbb{Z}}(X_\sigma, \mathbb{Z})$.
2. Let I_σ be the set of orbits of σ on I. For each element $\eta \in I_\sigma$, we set

$$\alpha_\eta := \begin{cases} \sum_{i\in\eta} \alpha_i & \text{if } a_{ij} = 0 \text{ for any two elements } i,\ j \text{ in } \eta \\ 2\sum_{i\in\eta} \alpha_i & \text{if } \eta = \{i, j\} \text{ and } a_{ij} = -1. \end{cases}$$

   Note that it covers all possible cases of η.
3. The embedding of X_σ into X induces a natural map $\theta : X^\vee \to X_\sigma^\vee$. Let $\alpha_\eta^\vee := \theta(\alpha_i^\vee)$ with i in η. Clearly $\alpha_\eta^\vee$ does not depend on the choice of i.

By [11, p. 29], $(X_\sigma^\vee, X_\sigma, \alpha_\eta^\vee, \alpha_\eta, \eta \in I_\sigma)$ is a root datum. It determines a reductive group G_σ. If G is simply-connected, then so is G_σ. Here is a table of G and G_σ for nontrivial σ [24, 6.4]:

1. If $G = A_{2n-1}$ and σ is of order 2, then $G_\sigma = B_n$, $n \geq 2$.
2. If $G = A_{2n}$ and σ is of order 2, then $G_\sigma = C_n$, $n \geq 1$.
3. If $G = D_n$ and σ is of order 2, then $G_\sigma = C_{n-1}$, $n \geq 4$.
4. If $G = D_4$ and σ is of order 3, then $G_\sigma = G_2$.
5. If $G = E_6$ and σ is of order 2, then $G_\sigma = F_4$.

Example 2.1. Let $G = \mathrm{SL}_{2n}$. We consider the automorphism

$$\sigma : G \longrightarrow G, \quad g \longmapsto \sigma(g) := w \cdot (g^t)^{-1} \cdot w^{-1},$$

where g^t is the transposition of g and $w = (w_{ij})$ is a matrix with entries

$$w_{ij} := \begin{cases} (-1)^j & \text{if } i+j = 2n+1 \\ 0 & \text{otherwise} \end{cases}.$$

The automorphism σ is a Dynkin automorphism on G. In this case, $G_\sigma = \mathrm{Spin}(2n+1)$.

2.3. The fixed point group G^σ

Let us fix a pinning $(T, B, x_i, y_i; i \in I)$ of G. Let σ be a Dynkin automorphism of G that preserves the pinning. Let G^σ be the identity component of the σ-fixed points of G. Let T^σ and B^σ be the identity components of the σ-fixed points of T and B respectively.

Recall the set I_σ of orbits of σ on I. For each orbit $\eta \in I_\sigma$, there are two cases:

1. If $a_{ij} = 0$ for any $i, j \in \eta$, then we set

$$x_\eta(a) := \prod_{i\in\eta} x_i(a), \quad y_\eta(a) := \prod_{i\in\eta} y_i(a), \quad \alpha_\eta^\vee := \sum_{i\in\eta} \alpha_i^\vee.$$

2. If $\eta = \{i, j\}$ and $a_{ij} = -1$, then we set

$$x_\eta(a) := x_i(a)x_j(2a)x_i(a), \quad y_\eta(a) := y_i(\frac{a}{2})y_j(a)y_i(\frac{a}{2}), \quad \alpha_\eta^\vee := 2(\alpha_i^\vee + \alpha_j^\vee).$$

Note that the definition of x_η and y_η does not depend on the ordering of elements in η.

Lemma 2.2. *The datum $(T^\sigma, B^\sigma, x_\eta, y_\eta; \eta \in I_\sigma)$ gives a pinning of G^σ.*

Proof. The first case is clear. The second case is due to a computation of SL_3. □

Remark 2.3. Let $G^\vee$ be the Langlands dual group of G. By considering the diagram automorphism σ on the dual root datum of $(X^\vee, X, \alpha_i^\vee, \alpha_i; i \in I)$, we get a Dynkin automorphism $\sigma^\vee$ of $G^\vee$. Let $(G^\vee)^{\sigma^\vee}$ be the identity component of the $\sigma^\vee$-fixed points of $G^\vee$. Note that the cocharacters of $(G^\vee)^{\sigma^\vee}$ are identified with the $\sigma^\vee$-invariant cocharacters of $G^\vee$. So $(G^\vee)^{\sigma^\vee}$ is the Langlands dual group of G_σ (see [20]).

Weyl groups of G and G^σ

Let s_i $(i \in I)$ be the simple reflections generating the Weyl group W of G. Set $\bar{s}_i := y_i(1)x_i(-1)y_i(1)$. The elements $\bar{s}_i$ satisfy the braid relations. So we can associate with each $w \in W$ its representative $\overline{w}$ in such a way that for any reduced decomposition $w = s_{i_1} \cdots s_{i_k}$ one has $\overline{w} = \bar{s}_{i_1} \cdots \bar{s}_{i_k}$. Let w_0 be the longest element of the Weyl group. Set $s_G := \overline{w}_0^2$. Note that s_G is a central element in G. Moreover $s_G^2 = 1$.

The Weyl group W^σ of G^σ can be naturally embedded into W with generators

$$s_\eta = \begin{cases} \prod_{i\in\eta} s_i, & \text{if } a_{ij}=0,\ \forall i,j\in\eta; \\ s_i s_j s_i, & \text{if } \eta=\{i,j\},\ a_{ij}=-1. \end{cases} \tag{9}$$

The longest element w_0 of W coincides with the longest element of W^σ. We state the following well-known fact for future use.

Lemma 2.4. *Each reduced decomposition $s_{\eta_1}\cdots s_{\eta_m}$ of w_0 in W^σ determines a reduced decomposition of w_0 in W with s_{η_i} expressed by (9), once we fix an ordering of elements in each η.*

Example 2.5. If the pair (G, G^σ) is of Cartan–Killing type (A_4, B_2), then

$$w_0 = s_{\eta_1}s_{\eta_2}s_{\eta_1}s_{\eta_2} = s_1s_4\cdot s_2s_3s_2\cdot s_1s_4\cdot s_2s_3s_2.$$

We set $\hat{s}_\eta := y_\eta(1)x_\eta(-1)y_\eta(1)$ for $\eta\in I_\sigma$. There is another representative $\bar{s}_\eta$ of s_η obtained by its decomposition in W. A direct calculation shows that $\hat{s}_\eta = h_\eta\bar{s}_\eta$, where $h_\eta = \alpha_i^\vee(2)\alpha_j^\vee(2)$ if $\eta=\{i,j\}, a_{ij}=-1$, and $h_\eta=1$ otherwise. We associate with w_0 a representative $\hat{w}_0$ via a reduced decomposition of w_0 in W^σ. Then

$$\hat{w}_0 = h\cdot\overline{w}_0, \qquad \text{where } h := \begin{cases} 1 & \text{if } G\neq A_{2n} \\ \prod_{k=1}^{n}(\alpha_k^\vee(2)\alpha_{2n+1-k}^\vee(2))^k & \text{if } G=A_{2n}. \end{cases} \tag{10}$$

Note that $s_G = s_{G^\sigma} := \hat{w}_0^2$.

3. Configuration space of decorated flags and its tropicalization

In this section, let us assume that G is defined over $\mathbb{Q}$. Let us fix a pinning $(T, B, x_i, y_i; i\in I)$ of G. Let σ be a Dynkin automorphism of G that preserves the pinning.

3.1. Positive spaces and their tropical points

Below we briefly introduce the category of positive spaces and the tropicalization functor.

Positive spaces

A positive rational function on a split algebraic torus $\mathcal{T}$ is a nonzero rational function on $\mathcal{T}$ which in a coordinate system, given by a set of characters of $\mathcal{T}$, can be presented as a ratio of two polynomials with positive integral coefficients. Denote by $\mathbb{Q}_+(\mathcal{T})$ the set of positive functions on $\mathcal{T}$.

A positive structure on an irreducible space (i.e., variety/stack) $\mathcal{Y}$ is a birational map γ from $\mathcal{T}$ to $\mathcal{Y}$. A rational function f on $\mathcal{Y}$ is called positive if $f\circ\gamma\in\mathbb{Q}_+(\mathcal{T})$. Denote by

$\mathbb{Q}_+(\mathcal{Y})$ the set of positive functions on $\mathcal{Y}$. Two positive structures on $\mathcal{Y}$ are equivalent if they determine the same set $\mathbb{Q}_+(\mathcal{Y})$. Such a pair $(\mathcal{Y}, \mathbb{Q}_+(\mathcal{Y}))$ is called a positive space.

Let $(\mathcal{Y}, \mathbb{Q}_+(\mathcal{Y}))$ and $(\mathcal{Z}, \mathbb{Q}_+(\mathcal{Z}))$ be a pair of positive spaces. A rational map $\phi : \mathcal{Y} \to \mathcal{Z}$ is called a positive map if $f \circ \phi \in \mathbb{Q}_+(\mathcal{Y})$ for all $f \in \mathbb{Q}_+(\mathcal{Z})$.

Tropicalization

Let $\big(\mathcal{Y}, \mathbb{Q}_+(\mathcal{Y})\big)$ be a positive space. A tropical point of $\mathcal{Y}$ is a map $l : \mathbb{Q}_+(\mathcal{Y}) \to \mathbb{Z}$ such that

$$\forall f, g \in \mathbb{Q}_+(\mathcal{Y}), \quad l(f+g) = \min\{l(f), l(g)\}, \quad l(fg) = l(f) + l(g).$$

Denote by $\mathcal{Y}(\mathbb{Z}^t)$ the set of tropical points of $\mathcal{Y}$. Tautologically, each $f \in \mathbb{Q}_+(\mathcal{Y})$ determines a $\mathbb{Z}$-valued function f^t of $\mathcal{Y}(\mathbb{Z}^t)$ such that $f^t(l) := l(f)$.

The following lemma is an easy exercise.

Lemma 3.1. *Let $\phi : \mathcal{Y} \to \mathcal{Z}$ be a positive map. There exists a unique map $\phi^t : \mathcal{Y}(\mathbb{Z}^t) \to \mathcal{Z}(\mathbb{Z}^t)$, called the tropicalization of ϕ, such that $(f \circ \phi)^t = f^t \circ \phi^t$ for all $f \in \mathbb{Q}_+(\mathcal{Z})$.*

The following lemma is standard. It shows that the tropicalization is a functor from the category of positive spaces to the category of the sets of tropical points.

Lemma 3.2. *Let $\phi : \mathcal{X} \to \mathcal{Y}$ and $\psi : \mathcal{Y} \to \mathcal{Z}$ be two positive maps. Then $(\psi \circ \phi)^t = \psi^t \circ \phi^t$.*

Let $f, g \in \mathbb{Q}_+(\mathcal{Y})$. We say $f < g$ if $g - f$ is still a positive function on $\mathcal{Y}$.

Lemma 3.3. *Let $f, g \in \mathbb{Q}_+(\mathcal{Y})$. If there exists a positive integer N such that $f < g < Nf$, then $f^t = g^t$.*

Proof. If $h := g - f \in \mathbb{Q}_+(\mathcal{Y})$, then $g^t = \min\{h^t, f^t\} \leq f^t$. Therefore $(Nf)^t \leq g^t \leq f^t$. Note that $(Nf)^t = f^t$. Therefore $g^t = f^t$. $\square$

Example 3.4. Denote by $X_*(\mathcal{T})$ and $X^*(\mathcal{T})$ the lattices of cocharacters and characters of a split algebraic torus $\mathcal{T}$. There is a perfect pairing

$$\langle\ ,\ \rangle : X_*(\mathcal{T}) \times X^*(\mathcal{T}) \to \mathbb{Z}.$$

Each $f \in \mathbb{Q}_+(\mathcal{T})$ can be presented as

$$f = \frac{\sum_{\alpha \in X^*(\mathcal{T})} c_\alpha X^\alpha}{\sum_{\alpha \in X^*(\mathcal{T})} d_\alpha X^\alpha}, \quad c_\alpha, d_\alpha \in \mathbb{N} = \{0, 1, 2, \cdots\}.$$

Here X^α is the regular function on $\mathcal{T}$ associated with α and c_α, d_α are zero for all but finitely many α. Each cocharacter $l \in X_*(\mathcal{T})$ determines a tropical point of $\mathcal{T}$ such that

$$l(f) := \min_{\alpha \mid c_\alpha \neq 0} \langle l, \alpha \rangle - \min_{\alpha \mid d_\alpha \neq 0} \langle l, \alpha \rangle.$$

It is easy to show that all tropical points of $\mathcal{T}$ can be defined this way. Therefore the set $\mathcal{T}(\mathbb{Z}^t)$ is canonically identified with $X_*(\mathcal{T})$. We treat them as the same set in this paper.

Lemma 3.5. *If $f \in \mathbb{Q}_+(\mathcal{T})$ is a regular function*[1] *on $\mathcal{T}$, then f^t is convex, i.e.,*

$$f^t(l_1 + l_2) \geq f^t(l_1) + f^t(l_2), \quad \forall l_1, l_2 \in X_*(\mathcal{T}).$$

Proof. The function f is a Laurent polynomial on $\mathcal{T}$:

$$f = \sum_{\alpha \in X^*(\mathcal{T})} c_\alpha X^\alpha, \quad c_\alpha \in \mathbb{Z}.$$

It is easy to show that $f^t(l) = \min_{\alpha | c_\alpha \neq 0} \langle l, \alpha \rangle$. The convexity follows. □

Lemma 3.6. *Let $\phi : \mathcal{T}_1 \to \mathcal{T}_2$ be a positive map between two split algebraic tori. If ϕ is regular, then $\phi^t : X_*(\mathcal{T}_1) \to X_*(\mathcal{T}_2)$ is linear.*

Proof. Let us write the map ϕ in coordinates:

$$\phi : \mathcal{T}_1 \longrightarrow \mathcal{T}_2, \quad x := (x_1, \ldots, x_n) \longmapsto (\phi_1(x), \ldots, \phi_m(x)).$$

If ϕ is a regular map, then every $\phi_i(x)$ is invertible. Therefore $\phi_i(x)$ must be monomials of $x_1, \ldots, x_n$ with nontrivial coefficients. So its tropicalization is linear. □

If the space $\mathcal{Y}$ admits a positive structure defined by a birational map $\gamma : \mathcal{T} \to \mathcal{Y}$, then γ^t is a bijection from $\mathcal{T}(\mathbb{Z}^t)$ to $\mathcal{Y}(\mathbb{Z}^t)$. For $l \in \mathcal{Y}(\mathbb{Z}^t)$, its pre-image $\beta(l) := (\gamma^t)^{-1}(l)$ is called the coordinate of l in $\mathcal{T}(\mathbb{Z}^t)$. Note that $\mathcal{T}(\mathbb{Z}^t) = X_*(\mathcal{T})$ is an abelian group, it induces an extra operation $+_\gamma$ on $\mathcal{Y}(\mathbb{Z}^t)$ such that

$$\beta(l +_\gamma l') = \beta(l) + \beta(l'), \quad l, l' \in \mathcal{Y}(\mathbb{Z}^t). \tag{11}$$

3.2. Lusztig's positive atlas of U_*

Let $U = [B, B]$ be the maximal unipotent subgroup inside B. Let B^- be the Borel subgroup such that $B \cap B^- = T$. Let $U_* = U \cap B^- w_0 B^-$.

Let $w_0 = s_{i_1} s_{i_2} \ldots s_{i_N}$ be a reduced decomposition in W. The sequence $\mathbf{i} = (i_1, \ldots, i_N)$ is called a reduced word for w_0 in W. There is an open embedding

$$\gamma_{\mathbf{i}} : \mathbb{G}_m^N \hookrightarrow U_*, \quad (a_1, \ldots, a_N) \longmapsto x_{i_1}(a_1) \ldots x_{i_N}(a_N). \tag{12}$$

[1] For example, $f = \frac{1+X^3}{1+X} = 1 - X + X^2$ is such a function on $\mathbb{G}_m$.

The birational map $\gamma_{\mathbf{i}}$ defines a positive structure of U_*. It is shown in [23] that all the reduced words for w_0 give rise to the equivalent positive structures on U_*, which we call Lusztig's positive atlas.

Note that the Dynkin automorphism σ preserves B and B^-. So it preserves U_*.

Lemma 3.7. *The automorphism $\sigma : U_* \longrightarrow U_*$ is a positive map.*

Proof. Let $\mathbf{i} = (i_1, \dots, i_N)$ be a reduced word for w_0. Then $\sigma(\mathbf{i}) = (\sigma(i_1), \dots, \sigma(i_N))$ is also a reduced word for w_0. For each $u = x_{i_1}(a_1)\dots x_{i_N}(a_N) \in U_*$, we have

$$\sigma(u) = x_{\sigma(i_1)}(a_1)\dots x_{\sigma(i_N)}(a_N) \in U_*.$$

Since the positive structures given by $\mathbf{i}$ and $\sigma(\mathbf{i})$ are equivalent, the lemma follows. □

The tropicalization of σ is a bijection

$$\sigma^t : U_*(\mathbb{Z}^t) \xrightarrow{\sim} U_*(\mathbb{Z}^t). \tag{13}$$

Denote by $\big(U_*(\mathbb{Z}^t)\big)^\sigma$ the set of σ^t-fixed points. Below we give a characterization of the σ^t-fixed points.

Let $\mathbf{j} = (\eta_1, \dots, \eta_n)$ be a reduced word for w_0 in W^σ. It determines a reduced word $\mathbf{i} = (i_1, i_2, \dots, i_N)$ for w_0 in W (Lemma 2.4). The tropicalization of (12) is a bijection

$$\gamma_{\mathbf{i}}^t : \mathbb{Z}^N \xrightarrow{=} U_*(\mathbb{Z}^t).$$

Denote by $(m_1, \dots, m_N)$ the pre-image of $l \in U_*(\mathbb{Z}^t)$ in $\mathbb{Z}^N$, which is called the tropical coordinate of l provided by $\gamma_{\mathbf{i}}$.

The following lemma is a manifestation of Proposition 3.5 in [10].

Lemma 3.8. *A tropical point l is σ^t-invariant if and only if*

$$m_1 = m_2 = \dots = m_{r_{\eta_1}}, \quad m_{r_{\eta_1}+1} = m_{r_{\eta_1}+2} = \dots = m_{r_{\eta_1}+r_{\eta_2}}, \quad \dots,$$

where r_η is the cardinality of the orbit η.

Proof. First we prove the case when G is of type A_2 and σ is of order 2. In this case, the set $I = \{1, 2\}$, and $\sigma(1) = 2$, $\sigma(2) = 1$. So $\mathbf{i} = (1, 2, 1)$ is a reduced word of w_0 in W. If $u = x_1(a)x_2(b)x_1(c)$, then

$$\sigma(u) = x_2(a)x_1(b)x_2(c) = x_1(\frac{bc}{a+c})x_2(a+c)x_1(\frac{ab}{a+c}).$$

Let (m_1, m_2, m_3) be the coordinate of l. So the coordinate of $\sigma^t(l)$ is

$$(m_2 + m_3 - \min\{m_1, m_3\}, \min\{m_1, m_3\}, m_1 + m_2 - \min\{m_1, m_3\}). \tag{14}$$

Note that $l = \sigma^t(l)$ if and only if $m_1 = m_2 = m_3$. The lemma follows.

The general case can be reduced to the above case and the case when G is of type $A_1 \times \cdots \times A_1$. The latter case follows by a similar but easier argument. $\square$

Let U^σ be the identity component of the σ-fixed points of U. The reduced word $\mathbf{j}$ of w_0 in W^σ determines a positive structure of U_*^σ:

$$\gamma_{\mathbf{j}} : \mathbb{G}_m^n \longrightarrow U_*^\sigma, \quad (a_1, \dots, a_n) \longmapsto x_{\eta_1}(a_1) \dots x_{\eta_n}(a_n).$$

Lemma 3.9. *The natural embedding $\imath : U_*^\sigma \hookrightarrow U_*$ is a positive map. The tropicalization of $\imath$ identifies the set $U_*^\sigma(\mathbb{Z}^t)$ with $(U_*(\mathbb{Z}^t))^\sigma$.*

Proof. Recall the construction of the pinning of G^σ in Section 2.3. It provides an explicit expression of the map $\imath$ using the coordinates provided by $\gamma_{\mathbf{j}}$ and $\gamma_{\mathbf{i}}$. Then the positivity of $\imath$ is clear. Then second part follows directly from Lemma 3.8. $\square$

The additive Whittaker character χ

The pinning of G determines an additive character χ of U such that

$$\chi(x_i(a)) = a, \quad \forall i \in I; \qquad \chi(u_1 u_2) = \chi(u_1) + \chi(u_2), \quad \forall u_1, u_2 \in U. \tag{15}$$

The following lemma is clear.

Lemma 3.10. *The restriction of χ on U_* is a positive function. The function χ is invariant under the automorphism σ, i.e., $\chi \circ \sigma = \chi$.*

Denote by χ_σ the additive Whittaker character of U^σ such that $\chi_\sigma(x_\eta(a)) = a$ for $\eta \in I_\sigma$, and $\chi_\sigma(u_1u_2) = \chi_\sigma(u_1) + \chi_\sigma(u_2)$ for $u_1, u_2 \in U^\sigma$. The restriction of χ_σ on U_*^σ is a positive function.

Lemma 3.11. *We have $\chi_\sigma^t = \chi^t \circ \imath^t$.*

Proof. It is easy to check that $\chi \circ \imath(x_\eta(a)) = \kappa_\eta \chi_\sigma(x_\eta(a))$, where

$$\kappa_\eta = \begin{cases} 1 & \text{if } \eta = \{i\}. \\ 2 & \text{if } \eta = \{i,j\}, \text{ and } a_{ij} = 0. \\ 3 & \text{if } \eta = \{i,j,k\}, \text{ and } a_{ij} = a_{jk} = a_{ik} = 0. \\ 4 & \text{if } \eta = \{i,j\}, \text{ and } a_{ij} = -1. \end{cases}$$

Hence in any case, $\chi_\sigma \le \chi \circ \imath \le 4\chi_\sigma$. By Lemma 3.3, $\chi_\sigma^t = (\chi \circ \imath)^t$. By Lemma 3.2, $(\chi \circ \imath)^t = \chi^t \circ \imath^t$. It concludes the proof of our lemma. $\square$

3.3. Configuration space of decorated flags

Let $\mathcal{A} := G/U$. The elements of $\mathcal{A}$ are called decorated flags. The group G acts on $\mathcal{A}$ on the left. For each $A \in \mathcal{A}$, the stabilizer $stab_G(A)$ is a maximal unipotent subgroup of G. Let π be the natural projection from $\mathcal{A}$ to the flag variety $\mathcal{B}$ such that $\pi(A)$ is the Borel subgroup containing $stab_G(A)$. It is easy to show that for each $B \in \mathcal{B}$, its fiber $\pi^{-1}(B)$ is a T-torsor.

We consider the *configuration space*

$$\mathrm{Conf}_n(\mathcal{A}) := G\backslash\mathcal{A}^n. \tag{16}$$

We say a pair $(B_1, B_2) \in \mathcal{B}^2$ is *generic* if $B_1 \cap B_2$ is a maximal torus of G. We consider the following open subspace of $\mathrm{Conf}_n(\mathcal{A})$:

$$\mathrm{Conf}_n^\times(\mathcal{A}) := \{G\backslash(A_1, \dots, A_n) \mid (\pi(A_i), \pi(A_{i+1})) \text{ is generic for each } i \in \mathbb{Z}/n\}. \tag{17}$$

Below we introduce a positive structure on $\mathrm{Conf}_n^\times(\mathcal{A})$, following [4, Section 8]. We also refer the readers to [8, Section 6] for more details.

We consider the space

$$\mathcal{R} := G\backslash\{(B_1, A, B_2) \mid (\pi(A), B_1), (\pi(A), B_2) \text{ are generic}\} \subset G\backslash(\mathcal{A} \times \mathcal{B}^2). \tag{18}$$

Denote by $\mathcal{R}_*$ the open subspace of $\mathcal{R}$ with requiring the pair (B_1, B_2) is also generic. Abusing notation, denote by U the decorated flag corresponding to the coset of the identity in $\mathcal{A}$.

Lemma 3.12. *(See [4, Section 8].) There is an isomorphism* $\mathbf{ed} : \mathrm{Conf}_2^\times(\mathcal{A}) \xrightarrow{\sim} T$ *such that*

$$(A_1, A_2) = (U, \mathbf{ed}(A_1, A_2)\overline{w}_0 \cdot U).$$

There is an isomorphism $\mathbf{an} : \mathcal{R} \xrightarrow{\sim} U$ *such that*

$$(B_1, A, B_2) = (B^-, U, \mathbf{an}(B_1, A, B_2) \cdot B^-).$$

The restriction of $\mathbf{an}$ *on* $\mathcal{R}_*$ *is an isomorphism from* $\mathcal{R}_*$ *to* U_**.*

Lemma–Construction 3.13. *(See [4, Section 8].) There is a natural open embedding*[2]

$$p : T^{n-1} \times U_*^{n-2} \hookrightarrow \mathrm{Conf}_n^\times(\mathcal{A}), \ (h_2, \dots, h_n, u_2, \dots, u_{n-1}) \longmapsto (A_1, \dots, A_n) \tag{19}$$

such that

[2] In fact, the images of p consist of configurations $(A_1, \dots, A_n) \in \mathrm{Conf}_n^\times(\mathcal{A})$ such that the pairs $(\pi(A_1), \pi(A_i))$ are also generic for $i = 2, \dots, n$.

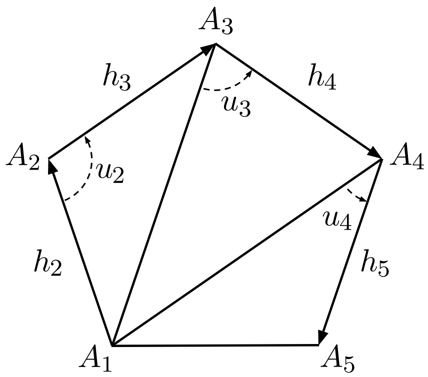

Fig. 1. The invariants assigned to $\mathrm{Conf}_5^\times(\mathcal{A})$.

- $h_i = \mathbf{ed}(A_{i-1}, A_i),\ i \in \{2, \ldots, n\}$;
- $u_j = \mathbf{an}(\pi(A_1), A_j, \pi(A_{j+1})),\ j \in \{2, \ldots, n-1\}$.

Let P_n be a convex n-gon. Let us assign to each vertex of P_n a decorated flag A_i so that $A_1, \ldots, A_n$ sit clockwise in the polygon. Then h_i, u_j are variables assigned to the edges and angles of P_n. See Fig. 1.

The positive structure on U_* is defined via Lusztig's atlas. Note that T is a split algebraic group and therefore admits a natural positive structure. So $T^{n-1} \times U_*^{n-2}$ admits a positive structure. From now on, we fix a positive structure on $\mathrm{Conf}_n^\times(\mathcal{A})$ such that the map p and its inverse p^{-1} are both positive maps.

Fock and Goncharov [4, Definition 2.5] defined the *twisted cyclic shift map*

$$r : \mathrm{Conf}_n^\times(\mathcal{A}) \longrightarrow \mathrm{Conf}_n^\times(\mathcal{A}), \quad (A_1, \ldots, A_n) \longmapsto (s_G \cdot A_n, A_1, \ldots, A_{n-1}). \tag{20}$$

They showed that

Theorem 3.14. *(See [4, Corollary 8.1].) The twisted cyclic shift map (20) is a positive map.*

Corollary 3.15. *The following map is a regular positive map:*

$$\mathbf{Ed} : \mathrm{Conf}_n^\times(\mathcal{A}) \longrightarrow T^n, \tag{21}$$

$$(A_1, \ldots, A_n) \longmapsto \big(\mathbf{ed}(s_G \cdot A_n, A_1), \mathbf{ed}(A_1, A_2), \ldots, \mathbf{ed}(A_{n-1}, A_n)\big).$$

Proof. The positivity of the first factor follows from Theorem 3.14. The rest is clear. $\square$

Recall the additive Whittaker character χ of U in Section 3.2.

Definition 3.16. (See [8, Section 2.1.4].) The *potential* $\mathcal{W}$ is a regular function of $\mathrm{Conf}_n^{\times}(\mathcal{A})$ such that

$$\mathcal{W}(A_1,\ldots,A_n) := \sum_{i\in\mathbb{Z}/n} \chi\big(\mathbf{an}(\pi(A_{i-1}),A_i,\pi(A_{i+1}))\big).$$

Corollary 3.17. *The potential $\mathcal{W}$ is a positive function.*

Proof. Note that $\mathbf{an}(\pi(A_1),A_2,\pi(A_3)\big)$ is a part of the map (19). By Lemma 3.10, χ is a positive function on U_*. So the summand $\chi\big(\mathbf{an}(\pi(A_1),A_2,\pi(A_3)\big)$ is a positive function. The central element s_G is contained in the intersection of Borel subgroups. Therefore

$$\mathbf{an}(B_1,s_G\cdot A,B_2)=\mathbf{an}(B_1,A,B_2).$$

Using Theorem 3.14, the rest summands are positive functions. □

3.4. The automorphism σ of $\mathrm{Conf}_n^{\times}(\mathcal{A})$

The automorphism σ of G preserves U. Thus it descends to an automorphism of $\mathcal{A}$. Similarly, it descends to an automorphism of $\mathcal{B}$. Recall the projection π from $\mathcal{A}$ to $\mathcal{B}$. Clearly σ commutes with the projection $\pi(\sigma(A))=\sigma(\pi(A))$ for $A\in\mathcal{A}$.

Abusing notation, we consider the automorphism

$$\sigma:\quad \mathrm{Conf}_n^{\times}(\mathcal{A}) \xrightarrow{\sim} \mathrm{Conf}_n^{\times}(\mathcal{A}),\quad (A_1,\ldots,A_n)\longrightarrow(\sigma(A_1),\ldots,\sigma(A_n)). \tag{22}$$

Lemma 3.18. *The map σ commutes with the invariants in Lemma 3.12:*

$$\mathbf{ed}(\sigma(A_1),\sigma(A_2))=\sigma(\mathbf{ed}(A_1,A_2)), \tag{23}$$

$$\mathbf{an}(\sigma(B_1),\sigma(A),\sigma(B_2))=\sigma(\mathbf{an}(B_1,A,B_2)). \tag{24}$$

Proof. Let $A_1=U$ and let $A_2=\mathbf{ed}(A_1,A_2)\overline{w}_0\cdot U$. Note that σ preserves U and $\overline{w}_0$. Therefore $\sigma(A_1)=U$ and $\sigma(A_2)=\sigma(\mathbf{ed}(A_1,A_2))\overline{w}_0\cdot U$. The first identity follows. The second identity follows by the same argument. □

Lemma 3.19. *The automorphism (22) is a positive map.*

Proof. Recall the birational map p in (19). By Lemma 3.18, we have the isomorphism

$$p^{-1}\circ\sigma\circ p:\ T^{n-1}\times U_*^{n-2}\xrightarrow{\sim} T^{n-1}\times U_*^{n-2}, \tag{25}$$

$$(h_2,\ldots,h_n,u_2,\ldots,u_{n-1})\longmapsto(\sigma(h_2),\ldots,\sigma(h_n),\sigma(u_2),\ldots,\sigma(u_{n-1})).$$

By Lemma 3.7, it is a positive map. Since p and p^{-1} are both positive maps, the automorphism σ is positive. □

Lemma 3.20. *The automorphism (22) preserves the potential $\mathcal{W}$, i.e.,*

$$\mathcal{W}\big(\sigma(A_1,\ldots,A_n)\big) = \mathcal{W}(A_1,\ldots,A_n). \tag{26}$$

Proof. It follows from Lemma 3.18 and the fact that $\chi(\sigma(u)) = \chi(u)$. $\square$

By Example 3.4, the set $T(\mathbb{Z}^t)$ is canonically identified with the lattice $X^\vee$ of cocharacters of T. Let $\underline{\lambda} = (\lambda_1,\ldots,\lambda_n) \in (X^\vee)^n$. We set $\sigma(\underline{\lambda}) := (\sigma(\lambda_1),\ldots,\sigma(\lambda_n))$.

Definition 3.21. We define the following set of tropical points:

$$\mathbf{C}_{\underline{\lambda},G} := \{l \in \mathrm{Conf}_n^\times(\mathcal{A})(\mathbb{Z}^t) \mid \mathbf{Ed}^t(l) = \underline{\lambda},\ \mathcal{W}^t(l) \geq 0\}. \tag{27}$$

We call a tropical point $l \in \mathbf{C}_{\underline{\lambda},G}$ a G-lamination of weight $\underline{\lambda}$.

Lemma 3.22. *The tropicalization of (22) gives rise to a bijection*

$$\sigma^t : \mathbf{C}_{\underline{\lambda},G} \xrightarrow{\sim} \mathbf{C}_{\sigma(\underline{\lambda}),G}. \tag{28}$$

Remark 3.23. For $\underline{\lambda} = \sigma(\underline{\lambda})$, denote by $(\mathbf{C}_{\underline{\lambda},G})^\sigma$ the set of fixed points under (28).

Proof. Let $l \in \mathrm{Conf}_n^\times(\mathcal{A})(\mathbb{Z}^t)$. It suffices to prove that

$$\mathcal{W}^t(\sigma^t(l)) = \mathcal{W}^t(l), \qquad \mathbf{Ed}^t(\sigma^t(l)) = \sigma(\mathbf{Ed}^t(l)). \tag{29}$$

The first identity is due to Lemma 3.20. The second identity is due to (23). $\square$

3.5. The embedding $\imath$ from $\mathrm{Conf}_n^\times(\mathcal{A}^\sigma)$ to $\mathrm{Conf}_n^\times(\mathcal{A})$

Let $\mathcal{A}^\sigma := G^\sigma/U^\sigma$. By the same construction as (19), the pinning of G^σ in Lemma 2.2 determines an open embedding

$$p^\sigma : (T^\sigma)^{n-1} \times (U_*^\sigma)^{n-2} \hookrightarrow \mathrm{Conf}_n^\times(\mathcal{A}^\sigma). \tag{30}$$

It induces a positive structure on the latter space.

There is a natural embedding from $\mathcal{A}^\sigma$ to $\mathcal{A}$. It induces a natural embedding

$$\imath : \ \mathrm{Conf}_n^\times(\mathcal{A}^\sigma) \hookrightarrow \mathrm{Conf}_n^\times(\mathcal{A}). \tag{31}$$

Proposition 3.24. *The embedding (31) is a positive map. The tropicalization of (31) gives a bijection from $\mathrm{Conf}_n^\times(\mathcal{A}^\sigma)(\mathbb{Z}^t)$ to the set of σ^t-fixed points of $\mathrm{Conf}_n^\times(\mathcal{A})(\mathbb{Z}^t)$.*

Proof. We consider the following composition map:

$$\jmath: \quad (T^\sigma)^{n-1} \times (U_*^\sigma)^{n-2} \xrightarrow{p^\sigma} \mathrm{Conf}_n^\times(\mathcal{A}^\sigma) \xrightarrow{\iota} \mathrm{Conf}_n^\times(\mathcal{A}) \xrightarrow{p^{-1}} T^{n-1} \times U^{n-2}. \tag{32}$$

Precisely it is given by

$$\big(h_2, \ldots, h_n, u_2, \ldots, u_{n-1}\big) \longmapsto \big(h_2 h, \ldots, h_n h, u_2, \ldots, u_{n-1}\big),$$

where h is the element in T described in (10). The element h appears because we use $\hat{w}_0$ instead of $\overline{w}_0$ to define the isomorphism from $\mathrm{Conf}_2^\times(\mathcal{A}^\sigma)$ to T^σ. Clearly (32) is a positive map. Therefore (31) is a positive map.

Let us tropicalize the map (32). Note that h does not contribute to the tropicalization. Therefore we get an injection

$$\jmath^t: \quad (T^\sigma(\mathbb{Z}^t))^{n-1} \times (U_*^\sigma(\mathbb{Z}^t))^{n-2} \longrightarrow (T(\mathbb{Z}^t))^{n-1} \times (U_*(\mathbb{Z}^t))^{n-2}$$

$$(\lambda_2, \ldots, \lambda_n, l_2, \ldots, l_{n-1}) \longmapsto (\lambda_2, \ldots, \lambda_n, \imath^t(l_2), \ldots, \imath^t(l_{n-1})).$$

By Lemma 3.9, the image of $\jmath^t$ is precisely the set of σ^t-fixed points. Thus the map ι^t is a bijection from $\mathrm{Conf}_n^\times(\mathcal{A}^\sigma)(\mathbb{Z}^t)$ to the set of σ^t-fixed points of $\mathrm{Conf}_n^\times(\mathcal{A}^\sigma)(\mathbb{Z}^t)$. $\square$

Similarly, we have the following positive map/function:

$$\mathbf{Ed}_\sigma : \mathrm{Conf}_n^\times(\mathcal{A}^\sigma) \longrightarrow (T^\sigma)^n, \qquad \mathcal{W}_\sigma : \mathrm{Conf}_n^\times(\mathcal{A}^\sigma) \longrightarrow \mathbb{A}^1. \tag{33}$$

Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (T^\sigma(\mathbb{Z}^t))^n \subset (T(\mathbb{Z}^t))^n$. Then $\sigma(\underline{\lambda}) = \underline{\lambda}$. We set

$$\mathbf{C}_{\underline{\lambda}, G^\sigma} := \{l \in \mathrm{Conf}_n^\times(\mathcal{A}^\sigma)(\mathbb{Z}^t) \mid \mathbf{Ed}_\sigma^t(l) = \underline{\lambda},\ \mathcal{W}_\sigma^t(l) \geq 0\}. \tag{34}$$

Theorem 3.25. *The tropicalization of (31) gives rise to a canonical bijection between σ-invariant G-laminations and G^σ-laminations,*

$$\iota^t : \mathbf{C}_{\underline{\lambda}, G^\sigma} \xrightarrow{\sim} \big(\mathbf{C}_{\underline{\lambda}, G}\big)^\sigma. \tag{35}$$

Proof. It follows from Proposition 3.24 and the identities

$$\mathcal{W}_\sigma^t = \mathcal{W}^t \circ \iota^t, \qquad \mathbf{Ed}_\sigma^t = \mathbf{Ed}^t \circ \iota^t. \tag{36}$$

The first identity in (36) is due to Lemma 3.11. The second identity follows similarly. $\square$

3.6. Summation of tropical points

Let $\lambda \in X^\vee$. We set

$$S(\lambda) := \begin{cases} \lambda + \sigma(\lambda); & \text{if } G \text{ is not of type } A_{2n} \text{ and } \sigma \text{ is of order 2,} \\ \lambda + \sigma(\lambda) + \sigma(\sigma(\lambda)) & \text{if } \sigma \text{ is of order 3,} \\ \lambda + \sigma(\lambda) + \sigma\big(\lambda + \sigma(\lambda)\big) & \text{if } \sigma \text{ is of order 2 and } G \text{ is of type } A_{2n}. \end{cases} \tag{37}$$

It is easy to show that $S(\lambda)$ is σ-invariant. In particular, if λ is σ-invariant, then $S(\lambda) = c_\sigma \lambda$, where c_σ is described in (3).

Let us fix a reduced word $\mathbf{i}$ for w_0 in W induced by a reduced word $\mathbf{j}$ for w_0 in W^σ. By (11), the Lusztig atlas $\gamma_{\mathbf{i}}$ determines an operation on $U_*(\mathbb{Z}^t)$, which we denote by $+_{\mathbf{i}}$ for short. Let $l \in \mathrm{U}_*(\mathbb{Z}^t)$. We set

$$S_{\mathbf{i}}(l) := \begin{cases} l +_{\mathbf{i}} \sigma^t(l); & \text{if } G \text{ is not of type } A_{2n} \text{ and } \sigma \text{ is of order 2,} \\ l +_{\mathbf{i}} \sigma^t(l) +_{\mathbf{i}} \sigma^t \circ \sigma^t(l) & \text{if } \sigma \text{ is of order 3,} \\ l +_{\mathbf{i}} \sigma^t(l) +_{\mathbf{i}} \sigma^t\big(l +_{\mathbf{i}} \sigma^t(l)\big) & \text{if } \sigma \text{ is of order 2 and } G \text{ is of type } A_{2n}. \end{cases} \tag{38}$$

Lemma 3.26. *We have* $S_{\mathbf{i}}(l) \in \big(U_*(\mathbb{Z}^t)\big)^\sigma$.

Proof. We prove the case when G is of type A_2 and σ is of order 2. The other cases follow by a similar but easier argument.

Note that $\mathbf{i} = (1, 2, 1)$ is a reduced word of w_0 in W. Let (m_1, m_2, m_3) be the coordinate of l provided by $\gamma_{\mathbf{i}}$. The coordinate of $\sigma^t(l)$ is given by (14). Thus the coordinate of $l +_{\mathbf{i}} \sigma^t(l)$ is (n, m, n), where

$$n = m_1 + m_2 + m_3 - \min\{m_1, m_3\}, \qquad m = m_2 + \min\{m_1, m_3\}.$$

Using (14) again, the coordinate of $\sigma^t\big(l +_{\mathbf{i}} \sigma^t(l)\big)$ is (m, n, m). The coordinate of $S_{\mathbf{i}}(l)$ is

$$(m + n, m + n, m + n) = (m_1 + 2m_2 + m_3, m_1 + 2m_2 + m_3, m_1 + 2m_2 + m_3).$$

By Lemma 3.8, $S_{\mathbf{i}}(l)$ is σ^t-invariant. $\square$

Let $\mathcal{T} := T^{n-1} \times (\mathbb{G}_m^N)^{n-2}$. There is a chain of open embedding

$$\Phi_{\mathbf{i}}: \quad \mathcal{T} = T^{n-1} \times (\mathbb{G}_m^N)^{n-2} \xhookrightarrow{id \times \gamma_{\mathbf{i}}^{n-2}} T^{n-1} \times (U_*)^{n-2} \xhookrightarrow{p} \mathrm{Conf}_n^\times(\mathcal{A}).$$

Its tropicalization is a chain of bijections

$$\Phi_{\mathbf{i}}^t: \quad \mathcal{T}(\mathbb{Z}^t) \xrightarrow{=} \big(T(\mathbb{Z}^t)\big)^{n-1} \times (U_*(\mathbb{Z}^t))^{n-2} \xrightarrow{=} \mathrm{Conf}_n^\times(\mathcal{A})(\mathbb{Z}^t). \tag{39}$$

By (11), $\Phi_{\mathbf{i}}^t$ induces an operation on $\mathrm{Conf}_n^\times(\mathcal{A})(\mathbb{Z}^t)$, which is denoted by $+_{\mathbf{i}}$ for short.

Lemma 3.27. *Let $l, l' \in \mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t)$. We have*

$$\mathcal{W}^t(l +_{\mathbf{i}} l') \geq \mathcal{W}^t(l) + \mathcal{W}^t(l'), \tag{40}$$

$$\mathbf{Ed}^t(l +_{\mathbf{i}} l') = \mathbf{Ed}^t(l) + \mathbf{Ed}^t(l'). \tag{41}$$

Proof. Note that $\mathcal{W} \circ \Phi_{\mathbf{i}}$ is a regular function of $\mathcal{T}$. The inequality (40) follows from Lemma 3.5. Note that $\mathbf{Ed} \circ \Phi_{\mathbf{i}}$ is a regular map from the torus $\mathcal{T}$ to the torus T^n. The identity (41) follows from Lemma 3.6. □

Lemma–Construction 3.28. *Let $l \in \mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t)$. We set*

$$\Sigma_{\mathbf{i}}(l) := \begin{cases} l +_{\mathbf{i}} \sigma^t(l); & \text{if } G \text{ is not of type } A_{2n} \text{ and } \sigma \text{ is of order } 2, \\ l +_{\mathbf{i}} \sigma^t(l) +_{\mathbf{i}} \sigma^t \circ \sigma^t(l) & \text{if } \sigma \text{ is of order } 3, \\ l +_{\mathbf{i}} \sigma^t(l) +_{\mathbf{i}} \sigma^t\big(l + \sigma^t(l)\big) & \text{if } \sigma \text{ is of order } 2 \text{ and } G \text{ is of type } A_{2n}. \end{cases}$$

Then $\Sigma_{\mathbf{i}}(l)$ is σ^t-invariant.

Proof. Note that the second bijection of (39) is given by the tropicalization of (19)

$$p^t : \big(T(\mathbb{Z}^t)\big)^{n-1} \times \big(U_*(\mathbb{Z}^t)\big)^{n-2} \xrightarrow{=} \mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t).$$

For $l \in \mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t)$, we consider its pre-image

$$\beta(l) := (p^t)^{-1}(l) = (\lambda_2, \ldots, \lambda_n, l_2, \ldots, l_{n-1}) \in \big(T(\mathbb{Z}^t)\big)^{n-1} \times \big(U_*(\mathbb{Z}^t)\big)^{n-2}.$$

Using (37)–(38), we get

$$\beta\big(\Sigma_{\mathbf{i}}(l)\big) = \big(S(\lambda_2), \ldots, S(\lambda_n), S_{\mathbf{i}}(l_2), \ldots, S_{\mathbf{i}}(l_{n-1})\big).$$

By Lemma 3.18, we have

$$\beta\big(\sigma^t(\Sigma_{\mathbf{i}}(l))\big) = \big(\sigma^t(S(\lambda_2)), \ldots, \sigma^t(S(\lambda_n)), \sigma^t(S_{\mathbf{i}}(l_2)), \ldots, \sigma^t(S_{\mathbf{i}}(l_{n-1}))\big).$$

By Lemma 3.26, we have $\beta\big(\Sigma_{\mathbf{i}}(l)\big) = \beta\big(\sigma^t(\Sigma_{\mathbf{i}}(l))\big)$. Therefore $\Sigma_{\mathbf{i}}(l)$ is σ^t-invariant. □

Theorem 3.29. *If $\mathbf{C}_{\underline{\lambda},G}$ is nonempty, then $(\mathbf{C}_{S(\underline{\lambda}),G})^\sigma$ is nonempty.*

Proof. If $\mathbf{C}_{\underline{\lambda},G}$ is nonempty, then we pick an element $l \in \mathbf{C}_{\underline{\lambda},G}$. It suffices to show that $\Sigma_{\mathbf{i}}(l) \in \big(\mathbf{C}_{S(\underline{\lambda}),G}\big)^\sigma$. By Lemma 3.22 and Lemma 3.27, we get

$$\mathcal{W}^t(\Sigma_{\mathbf{i}}(l)) \geq c_\sigma \mathcal{W}^t(l) \geq 0, \qquad \mathbf{Ed}^t(\Sigma_{\mathbf{i}}(l)) = S(\mathbf{Ed}^t(l)) = S(\underline{\lambda}).$$

So $\Sigma_{\mathbf{i}}(l) \in \mathbf{C}_{S(\underline{\lambda}),G}$. By Lemma 3.28, $\Sigma_{\mathbf{i}}(l)$ is σ^t-invariant. □

Remark 3.30. When $\underline{\lambda}$ is a triple of dominants weights, it should also be possible to prove Theorem 3.29 by using Berenstein–Zelevinsky patterns or Mirković–Vilonen polytopes. But it seems to us that the combinatorics involved could be very tedious.

4. Affine Grassmannian and Satake basis

4.1. Top components of cyclic convolution variety

Let $\mathrm{Gr}_G := G(\mathcal{K})/G(\mathcal{O})$ be the affine Grassmannian of G. We consider the action of the maximal torus T on Gr_G. The fixed points of $T(\mathcal{O})$ on Gr_G consist of $[\lambda] = t^\lambda \cdot G(\mathcal{O})$, where λ is a coweight of G and $t^\lambda \in T(\mathcal{K})$.

Let $X_+^\vee$ denote the cone of dominant coweights of G. For $L_1, L_2 \in \mathrm{Gr}_G$, there exists a unique $\lambda \in X_+^\vee$ such that $G(\mathcal{K}) \cdot (L_1, L_2) = G(\mathcal{K}) \cdot ([0], [\lambda])$. We write $d(L_1, L_2) := \lambda$, which we call the distance from L_1 to L_2.

Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (X_+^\vee)^n$. We consider the *cyclic convolution variety*

$$\mathrm{Gr}_{G,c(\underline{\lambda})} := \{(L_1, L_2, \ldots, L_n) \mid L_n = [0];\ d(L_{i-1}, L_i) = \lambda_i \text{ for } i \in \mathbb{Z}/n\}. \tag{42}$$

The variety $\mathrm{Gr}_{G,c(\underline{\lambda})}$ is of (complex) dimension

$$\mathrm{ht}(\underline{\lambda}) := \langle \rho, \lambda_1 + \lambda_2 + \ldots + \lambda_n \rangle,$$

where ρ is the half sum of positive roots of G. Denote by $\mathbf{T}_{\underline{\lambda},G}$ the set of irreducible components of $\mathrm{Gr}_{G,c(\underline{\lambda})}$ of dimension $\mathrm{ht}(\underline{\lambda})$.

Note that the Dynkin automorphism σ of G preserves $G(\mathcal{O})$. So it descends to an automorphism of Gr_G. Clearly σ commutes with the distance map:

$$d(\sigma(L_1), \sigma(L_2)) = \sigma(d(L_1, L_2)), \qquad \forall L_1, L_2 \in \mathrm{Gr}_G. \tag{43}$$

Therefore we get a natural bijection

$$\sigma : \mathrm{Gr}_{G,c(\underline{\lambda})} \xrightarrow{\sim} \mathrm{Gr}_{G,c(\sigma(\underline{\lambda}))}, \qquad (L_1, L_2, \ldots, L_n) \longmapsto (\sigma(L_1), \sigma(L_2), \ldots, \sigma(L_n)).$$

It induces a bijection on the set of top components

$$\sigma : \mathbf{T}_{\underline{\lambda},G} \xrightarrow{\sim} \mathbf{T}_{\sigma(\underline{\lambda}),G}. \tag{44}$$

4.2. Parameterization of top components

First we briefly recall the *constructible functions* in [8, Section 2.2.5].

Let R be a reductive group over $\mathbb{C}$. Let $\mathcal{X}$ be a rational space over $\mathbb{C}$. We assume that there is a rational left algebraic action of R on $\mathcal{X}$. Let $\mathbb{C}(\mathcal{X})$ be the function field of $\mathcal{X}$. Denote by $\circ$ the induced right action of R on $\mathbb{C}(\mathcal{X})$:

$$\forall g \in R, \ \ \forall F \in \mathbb{C}(\mathcal{X}), \qquad \big(F \circ g\big)(x) := F(g \cdot x).$$

Let $\mathcal{K}(\mathcal{X})$ be the field of rational functions of $\mathcal{X}$ with $\mathcal{K}$-coefficients. The valuation of $\mathcal{K}^\times$ induces a natural valuation map

$$\mathrm{val}: \quad \mathcal{K}(\mathcal{X})^\times \longrightarrow \mathbb{Z}.$$

Here $R(\mathcal{K})$ acts on $\mathcal{K}(\mathcal{X})$ on the right. Each $F \in \mathcal{K}(\mathcal{X})^\times$ gives rise to a $\mathbb{Z}$-valued function

$$D_F : R(\mathcal{K}) \longrightarrow \mathbb{Z}, \qquad g \longmapsto \mathrm{val}(F \circ g). \tag{45}$$

The action of the subgroup $R(\mathcal{O})$ preserves the valuation of $\mathcal{K}(\mathcal{X})^\times$ [8, Lemma 2.21]. So D_F descends to a function from $R(\mathcal{K})/R(\mathcal{O})$ to $\mathbb{Z}$ which we also denotes by D_F.

Assume that there is an automorphism τ of R and an isomorphism τ of $\mathcal{X}$ such that

$$\tau(g \cdot x) = \tau(g) \cdot \tau(x), \qquad \forall g \in R,\ \forall x \in \mathcal{X}. \tag{46}$$

We define a field isomorphism $\tau : \mathbb{C}(\mathcal{X}) \xrightarrow{\sim} \mathbb{C}(\mathcal{X})$ such that

$$\forall F \in \mathbb{C}(\mathcal{X}), \quad \tau(F)(x) := F(\tau^{-1}(x)). \tag{47}$$

It is easy to check that

$$\forall g \in R, \ \ \forall F \in \mathbb{C}(\mathcal{X}), \qquad \tau(F \circ g) = \tau(F) \circ \tau(g). \tag{48}$$

Lemma 4.1. *We have $D_{\tau(F)}(\tau(g)) = D_F(g)$.*

Proof. Note that τ preserves the valuation of $\mathcal{K}(\mathcal{X})^\times$. Therefore

$$D_{\tau(F)}(\tau(g)) = \mathrm{val}\big(\tau(F) \circ \tau(g)\big) = \mathrm{val}\big(\tau(F \circ g)\big) = \mathrm{val}\big(F \circ g\big) = D_F(g). \qquad \square$$

Now let $\mathcal{X} := \mathcal{A}^n$ and let $R := G^n$. The group G^n acts on $\mathcal{A}^n$ on the left. Note that the set $\mathbb{Q}_+\big(\mathrm{Conf}_n^\times(\mathcal{A})\big)$ of positive functions of $\mathrm{Conf}_n^\times(\mathcal{A})$ is contained in $\mathcal{K}(\mathcal{A}^n)^\times$. Each $F \in \mathbb{Q}_+\big(\mathrm{Conf}_n(\mathcal{A})\big)$ induces a function

$$D_F : R(\mathcal{K})/R(\mathcal{O}) = \mathrm{Gr}_G^n \longrightarrow \mathbb{Z},$$

which we call a constructible function on Gr_G^n.

Theorem 4.2. *(See [8, Theorems 2.20, 2.23].) There is a canonical bijection*

$$\kappa : \mathbf{C}_{\underline{\lambda},G} \longrightarrow \mathbf{T}_{\underline{\lambda},G}, \qquad l \longmapsto \kappa(l), \tag{49}$$

such that for each $F \in \mathbb{Q}_+\big(\mathrm{Conf}_n(\mathcal{A})\big)$, the generic value of the constructible function D_F on the component $\kappa(l)$ is equal to $F^t(l)$.

Remark 4.3. Let $l, l' \in \mathrm{Conf}_n(\mathcal{A})(\mathbb{Z}^t)$. By the definition of tropical points, we have

$$l = l' \quad \Longleftrightarrow \quad F^t(l) = F^t(l'), \quad \forall F \in \mathbb{Q}_+\big(\mathrm{Conf}_n(\mathcal{A})\big).$$

Therefore to identify two top components in $\mathbf{T}_{\underline{\lambda},G}$, it suffices to show that the generic values of all constructible functions on both components are equal.

Recall the bijections (28) and (44).

Theorem 4.4. *The following diagram commutes:*

$$\begin{tikzcd} \mathbf{C}_{\underline{\lambda},G} \arrow[r, "\kappa"] \arrow[d, "\sigma^t"'] & \mathbf{T}_{\underline{\lambda},G} \arrow[d, "\sigma"] \\ \mathbf{C}_{\sigma(\underline{\lambda}),G} \arrow[r, "\kappa"] & \mathbf{T}_{\sigma(\underline{\lambda}),G} \end{tikzcd} .$$

Proof. Let σ be the automorphism of G^n such that $\sigma(g_1, \ldots, g_n) := (\sigma(g_1), \ldots, \sigma(g_n))$. Let σ be the isomorphism of $\mathcal{A}^n$ such that $\sigma(A_1, \ldots, A_n) := (\sigma(A_1), \ldots, \sigma(A_n))$. Note that $\sigma(g \cdot A) = \sigma(g) \cdot \sigma(A)$. So σ agrees with τ map in (46).

Let $F \in \mathbb{Q}_+\big(\mathrm{Conf}_n^{\times}(\mathcal{A})\big)$ and let $(L_1, \ldots, L_n) \in \mathrm{Gr}_{G,c(\underline{\lambda})}$. By Lemma 4.1, we have

$$D_F(\sigma(L_1), \ldots, \sigma(L_n)) = D_{\sigma^{-1}(F)}(L_1, \ldots, L_n).$$

Let $l \in \mathbf{C}_{\underline{\lambda},G}$. Then the generic value of D_F on $\sigma(\kappa(l))$ is equal to the generic value of $D_{\sigma^{-1}(F)}$ on $\kappa(l)$. By Theorem 4.2 and (47), the latter is $\big(\sigma^{-1}(F)\big)^t(l) = F^t(\sigma^t(l))$, which is the generic value of D_F on the component $\kappa(\sigma^t(l))$. By Remark 4.3, $\sigma(\kappa(l))$ and $\kappa(\sigma^t(l))$ are the same component. $\square$

Let $\underline{\lambda} = \sigma(\underline{\lambda})$. Denote by $(\mathbf{T}_{\underline{\lambda},G})^{\sigma}$ the set of σ-stable top components of $\mathrm{Gr}_{c(\underline{\lambda})}$.

Corollary 4.5. *There is a natural bijection between $(\mathbf{T}_{\underline{\lambda},G})^{\sigma}$ and $\mathbf{T}_{\underline{\lambda},G^{\sigma}}$.*

Proof. It follows from the following sequence of bijections:

$$(\mathbf{T}_{\underline{\lambda},G})^{\sigma} \simeq (\mathbf{C}_{\underline{\lambda},G})^{\sigma} \simeq \mathbf{C}_{\underline{\lambda},G^{\sigma}} \simeq \mathbf{T}_{\underline{\lambda},G^{\sigma}}.$$

The first bijection is due to Theorem 4.4. The second bijection is due to Theorem 3.25. The third bijection is due to Theorem 4.2. $\square$

4.3. Geometric Satake correspondence and Satake basis

Let $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ be the category of $G(\mathcal{O})$-equivariant perverse sheaves. Let $\mathrm{Rep}(G^{\vee})$ be the category of finite dimensional representations of $G^{\vee}$. The geometric

Satake correspondence (e.g. [25, Theorem 14.1]) asserts that there is an equivalence of tensor categories

$$\mathbb{H} : \mathrm{Perv}_{G^\vee(\mathcal{O})}(\mathrm{Gr}_G) \simeq \mathrm{Rep}(G^\vee),$$

where $\mathbb{H}$ is given by the hypercohomology of perverse sheaves. The tensor category structure on $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ can be defined via the convolution product (e.g. [25, Section 4]).

Let $\underline{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ be a sequence of dominant coweights of G. We define the *convolution variety*

$$\mathrm{Gr}_{G,\underline{\lambda}} := \{(L_1, L_2, \ldots, L_n) \mid d([0], L_1) = \lambda_1;\ d(L_{i-1}, L_i) = \lambda_i \text{ for } i = 2, \ldots, n\}. \tag{50}$$

Denote by $\overline{\mathrm{Gr}_{G,\underline{\lambda}}}$ the closure of $\mathrm{Gr}_{G,\underline{\lambda}}$ in $(\mathrm{Gr}_G)^n$. Let $\mathrm{IC}_{\underline{\lambda}}$ be the IC sheaf supported on $\overline{\mathrm{Gr}_{G,\underline{\lambda}}}$. There is a natural projection

$$p : (\mathrm{Gr}_G)^n \to \mathrm{Gr}_G, \qquad p(L_1, L_2, \cdots, L_n) = L_n. \tag{51}$$

The convolution products of perverse sheaves in $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ are defined such that

$$\mathrm{IC}_{\lambda_1} * \mathrm{IC}_{\lambda_2} * \cdots * \mathrm{IC}_{\lambda_n} = p_*(\mathrm{IC}_{\underline{\lambda}}), \tag{52}$$

where p_* is the pushforward of sheaves in the derived setting. Note that the cyclic convolution variety $\mathrm{Gr}_{G,c(\underline{\lambda})}$ is the fiber

$$\mathrm{Gr}_{G,c(\underline{\lambda})} = p^{-1}([0]) \cap \mathrm{Gr}_{G,\underline{\lambda}}.$$

Recall that $\mathrm{ht}(\underline{\lambda}) = \langle \rho, \sum_{i=1}^n \lambda_i \rangle$. We have

$$\dim \mathrm{Gr}_{G,\underline{\lambda}} = 2\mathrm{ht}(\underline{\lambda}), \qquad \dim \mathrm{Gr}_{G,c(\underline{\lambda})} = \mathrm{ht}(\underline{\lambda}).$$

The following lemma is well known (cf. [9, Prop. 3.1]).

Lemma 4.6. *There is a canonical isomorphism $\alpha : V_{\underline{\lambda}}^{G^\vee} \simeq H_{\mathrm{top}}(\mathrm{Gr}_{G,c(\underline{\lambda})}, \mathbb{C})$, where $H_{\mathrm{top}}(\mathrm{Gr}_{G,c(\underline{\lambda})}, \mathbb{C})$ is the top Borel–Moore homology of $\mathrm{Gr}_{G,c(\underline{\lambda})}$. As a consequence, the set of top components of $\mathrm{Gr}_{G,c(\underline{\lambda})}$ provides a basis of $V_{\underline{\lambda}}^{G^\vee}$.*

Let σ be a Dynkin automorphism of G. Note that σ preserves $G(\mathcal{O})$. Thus it descends to an action σ on Gr_G. By pulling back sheaves, we get an auto-functor σ^* of $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$. By the Tannakian formalism, there is an automorphism $\tilde{\sigma}$ of $G^\vee$, such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) \arrow[r, "\mathbb{H}"] \arrow[d, "\sigma^*"] & \mathrm{Rep}(G^\vee) \arrow[d, "(\tilde{\sigma})^*"] \\ \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) \arrow[r, "\mathbb{H}"] & \mathrm{Rep}(G^\vee) \end{tikzcd},$$

where $(\tilde{\sigma})^*$ is the composition functor $(\rho, V) \mapsto (\rho \circ \tilde{\sigma}, V)$ for any representation (ρ, V) of $G^\vee$.

The following lemma asserts that $\tilde{\sigma}$ is a Dynkin automorphism of $G^\vee$.

Lemma 4.7. *(See [10, Theorem 4.2].) The automorphism $\tilde{\sigma}$ on $G^\vee$ is a Dynkin automorphism arising from the automorphism σ on the root datum $(X^\vee, X, \alpha_i^\vee, \alpha_i; i \in I)$.*

Abusing notations, denote by $\tilde{\sigma}$ the actions on V_{λ_i}, $V_{\underline{\lambda}}$ and $V_{\underline{\lambda}}^{G^\vee}$ induced by the automorphism $\tilde{\sigma}$ on $G^\vee$.

Let $\sigma^\sharp$ be the action on $V_{\underline{\lambda}}^{G^\vee}$ induced by the interchange map on the components of $\mathrm{Gr}_{G,\underline{c}(\lambda)}$ via the natural isomorphism $\alpha : V_{\underline{\lambda}}^{G^\vee} \simeq H_{\mathrm{top}}(\mathrm{Gr}_{G,c(\underline{\lambda})}, \mathbb{C})$ as in Lemma 4.6.

Proposition 4.8. *The actions of $\tilde{\sigma}$ and $\sigma^\sharp$ on $V_{\underline{\lambda}}^{G^\vee}$ coincide.*

Proof. We consider the natural isomorphisms $\phi_i : \sigma^*\mathrm{IC}_{\lambda_i} \simeq \mathrm{IC}_{\lambda_i}$ which are compatible with the interchange action on cycles (see [10, Section 4]). Applying the hypercohomology $\mathbb{H}$, we get automorphisms $\mathbb{H}(\phi_i) : V_{\lambda_i} \simeq V_{\lambda_i}$. Lemma 4.1 in [10] shows that $\mathbb{H}(\phi_i)$ coincides with the action $\tilde{\sigma}$ on V_{λ_i}.

Recall that the convolution product in $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ can also be constructed as the fusion product of sheaves via Beilinson–Drinfeld Grassmannian [25, Section 5]. From this point of view, it is easy to see that the isomorphisms ϕ_i give rise to an isomorphism

$$\phi : \sigma^*(\mathrm{IC}_{\lambda_1} * \mathrm{IC}_{\lambda_2} * \cdots * \mathrm{IC}_{\lambda_n}) \simeq \mathrm{IC}_{\lambda_1} * \mathrm{IC}_{\lambda_2} * \cdots * \mathrm{IC}_{\lambda_n}.$$

Applying the functor $\mathbb{H}$, we get

$$\mathbb{H}(\phi) : V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_n} \simeq V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_n}.$$

By the proof in [25, Proposition 6.1] that $\mathbb{H}$ is a tensor functor, we see that $\mathbb{H}(\phi)$ coincides with the diagonal automorphism $\mathbb{H}(\phi_1) \otimes \mathbb{H}(\phi_2) \otimes \cdots \otimes \mathbb{H}(\phi_n)$. Hence $\mathbb{H}(\phi)$ coincides with the automorphism $\tilde{\sigma}$ on $V_{\underline{\lambda}} := V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_n}$.

Let $i_0 : \mathrm{pt} \to \mathrm{Gr}_G$ be the embedding such that $i_0(\mathrm{pt}) = [0]$. It is well known that

$$H^0((i_0)^! p_* \mathrm{IC}_{\underline{\lambda}}) \simeq H_{\mathrm{top}}(\mathrm{Gr}_{G,c(\underline{\lambda})}, \mathbb{C}),$$

where $(i_0)^!$ is the right adjoint functor of the pushforward functor $(i_0)_*$, and H^0 is the degree zero cohomology of complex of vector spaces.

From the counit of the adjunction between $(i_0)_*$ and $i_0^!$, there exists a natural morphism

$$\iota : (i_0)_*(i_0)^! p_* \mathrm{IC}_{\underline{\lambda}} \to p_* \mathrm{IC}_{\underline{\lambda}}. \tag{53}$$

Applying the hypercohomology $\mathbb{H}$, in view of Lemma 4.6 we get the natural inclusion $V_{\underline{\lambda}}^{G^\vee} \hookrightarrow V_{\underline{\lambda}}$.

Recall that $\mathrm{IC}_{\lambda_1} * \mathrm{IC}_{\lambda_2} * \cdots * \mathrm{IC}_{\lambda_n} = p_*(\mathrm{IC}_{\underline{\lambda}})$. Then $\mathbb{H}(p_*\mathrm{IC}_{\underline{\lambda}})$ is naturally identified with the intersection cohomology of $\overline{\mathrm{Gr}_{G,\underline{\lambda}}}$. There is a unique isomorphism $\tilde{\phi} : \sigma^*\mathrm{IC}_{\underline{\lambda}} \simeq \mathrm{IC}_{\underline{\lambda}}$, induced from the interchange action on cycles classes. By natural constructions of ϕ and $\tilde{\phi}$, the following diagram commutes:

$$\begin{tikzcd} p_*\sigma^*\mathrm{IC}_{\underline{\lambda}} \arrow[r, "p_*(\tilde{\phi})"] \arrow[d, "\theta"'] & p_*\mathrm{IC}_{\underline{\lambda}} \\ \sigma^*p_*\mathrm{IC}_{\underline{\lambda}} \arrow[ur, "\phi"'] & \end{tikzcd}, \tag{54}$$

where θ is given by the base-change isomorphism. Note that $\mathbb{H}(\theta)$ is the identity map on $\mathbb{H}(p_*\mathrm{IC}_{\underline{\lambda}})$ and $(i_0)^!(\theta)$ is the identity map on $(i_0)^!(p_*\mathrm{IC}_{\underline{\lambda}})$. Therefore $\mathbb{H}(p_*(\tilde{\phi})) = \mathbb{H}(\phi)$ and $(i_0)^!(p_*(\tilde{\phi})) = (i_0)^!(\phi)$. By the functoriality of the counit $(i_0)_*(i_0)^! \to id$, we have the following commutative diagram:

$$\begin{tikzcd} (i_0)_*(i_0)^!(p_*\sigma^*\mathrm{IC}_{\underline{\lambda}}) \arrow[r, "(i_0)_*(i_0)^!p_*(\tilde{\phi})"] \arrow[d, "\iota"] & (i_0)_*(i_0)^!(p_*\mathrm{IC}_{\underline{\lambda}}) \arrow[d, "\iota"] \\ p_*\sigma^*\mathrm{IC}_{\underline{\lambda}} \arrow[r, "p_*(\tilde{\phi})"] & p_*\mathrm{IC}_{\underline{\lambda}} \end{tikzcd} \tag{55}$$

Applying the hypercohomology $\mathbb{H}$ to this commutative diagram, we can see that the restriction of $\mathbb{H}(\phi) = \mathbb{H}(p_*\tilde{\phi})$ on $V_{\underline{\lambda}}^{G^\vee}$ coincides with the automorphism

$$V_{\underline{\lambda}}^{G^\vee} \simeq H^0\big((i_0)^! p_*\mathrm{IC}_{\underline{\lambda}}\big) \xrightarrow{(i_0)^!(p_*\tilde{\phi})} H^0\big((i_0)^! p_*\mathrm{IC}_{\underline{\lambda}}\big) \simeq V_{\underline{\lambda}}^{G} \cdot$$

The map $(i_0)^!(p_*\tilde{\phi})$ interchanges the homology classes given by the top components of $\mathrm{Gr}_{G,c(\underline{\lambda})}$ in $H_{\mathrm{top}}(\mathrm{Gr}_{G,c(\underline{\lambda})}, \mathbb{C})$. Hence the proposition follows. □

5. Proof of main results

5.1. Proof of Theorem 1.1

Let σ be the given Dynkin automorphism of G. It induces an automorphism σ of the root datum $(X^\vee, X, \alpha_i^\vee, \alpha_i; i \in I)$. Further, we get an associated automorphism $\sigma^\vee$ of the dual datum $(X, X^\vee, \alpha_i, \alpha_i^\vee; i \in I)$. Abusing notation, denote by $\sigma^\vee$ the Dynkin automorphism of $G^\vee$ arising from the diagram automorphism $\sigma^\vee$.

As explained as in Section 4.3, by Tannakian formalism, the Dynkin automorphism $\sigma^\vee$ on $G^\vee$ induces a Dynkin automorphism $\tilde{\sigma}$ on G, which is compatible with the Satake basis (Proposition 4.8).

Lemma 5.1. *Let σ_1 and σ_2 be Dynkin automorphisms of G that induce the same diagram automorphism of root datum of G. Denote by σ_1 and σ_2 the induced actions on $V_{\underline{\lambda}}^G$ respectively. We have*

$$\text{trace } (\sigma_1 : V_{\underline{\lambda}}^G \to V_{\underline{\lambda}}^G) = \text{trace } (\sigma_2 : V_{\underline{\lambda}}^G \to V_{\underline{\lambda}}^G).$$

Proof. Assume that σ_1 preserves a pinning $(B, T, x_i, y_i; i \in I)$ and σ_2 preserves another pinning $(B, T, x_i', y_i'; i \in I)$. Let ψ be the automorphism of G such that its restriction on T is an identity map and $\psi(x_i(a)) = x_i'(a)$, $\psi(y_i(a)) = y_i'(a)$. By isomorphism theorem of reductive groups, ψ is an inner automorphism of G. Clearly $\sigma_2 = \psi \circ \sigma_1 \circ \psi^{-1}$. Note that the induced actions ψ and ψ^{-1} on $V_{\underline{\lambda}}$ preserve $V_{\underline{\lambda}}^G$. Hence the lemma follows. $\square$

By Lemma 5.1, the Dynkin automorphism σ of G can be replaced by the automorphism $\tilde{\sigma}$. By Proposition 4.8, the trace of σ on $V_{\underline{\lambda}}^G$ is equal to the number of $\sigma^\vee$-stable top components of $\mathrm{Gr}_{G^\vee, c(\underline{\lambda})}$. By Lemma 4.6, the dimension of $W_{\underline{\lambda}}^{G_\sigma}$ is equal to the number of the top components of $\mathrm{Gr}_{(G_\sigma)^\vee, c(\underline{\lambda})}$. Note that $(G_\sigma)^\vee$ is isomorphic to the identity component group of the $\sigma^\vee$-fixed points in $G^\vee$ (Remark 2.3). To summarize, we have the following sequence:

$$\text{trace}(\sigma : V_{\underline{\lambda}}^G \to V_{\underline{\lambda}}^G) \overset{\text{Proposition 4.8}}{=} \#(\mathbf{T}_{\underline{\lambda}, G^\vee})^{\sigma^\vee} \overset{\text{Corollary 4.5}}{=} \#\mathbf{T}_{\underline{\lambda}, (G_\sigma)^\vee} \overset{\text{Lemma 4.6}}{=} \dim W_{\underline{\lambda}}^{G_\sigma}.$$

Theorem 1.1 is proved.

5.2. Proof of Theorem 1.2

By Theorem 4.2 and Lemma 4.6, the dimension of the tensor invariant space $V_{\underline{\lambda}}^G$ equals the cardinality of the set $\mathbf{C}_{\underline{\lambda}, G^\vee}$. Therefore G has saturation factor k if and only if

- for any sequence $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ of dominant weights of G such that $\sum_{i=1}^n \lambda_i$ is in the root lattice of G, if $\mathbf{C}_{N\underline{\lambda}, G^\vee}$ is nonempty for some positive integer N, then $\mathbf{C}_{k\underline{\lambda}, G^\vee}$ is nonempty.

Now we prove Theorem 1.2. Let $\underline{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ be a sequence of dominant weights of G_σ such that $\sum_{i=1}^n \lambda_i$ is in the root lattice of G_σ. Assume that there is a positive integer N such that $\mathbf{C}_{N\underline{\lambda}, (G_\sigma)^\vee}$ is nonempty. It remains to show that $\mathbf{C}_{kc_\sigma \cdot \underline{\lambda}, (G_\sigma)^\vee}$ is nonempty.

Note that $(G_\sigma)^\vee$ is the identity component of the $\sigma^\vee$-fixed points of $G^\vee$. Theorem 3.25 implies that $(\mathbf{C}_{N\underline{\lambda}, G^\vee})^{\sigma^\vee}$ is nonempty. Therefore $\mathbf{C}_{N\underline{\lambda}, G^\vee}$ is nonempty. By the assumption that G has saturation factor k, we know that $\mathbf{C}_{k\underline{\lambda}, G^\vee}$ is nonempty. Theorem 3.29 implies that $(\mathbf{C}_{kc_\sigma \cdot \underline{\lambda}, G^\vee})^{\sigma^\vee}$ is nonempty. Using Theorem 3.25 again, the set $\mathbf{C}_{kc_\sigma \cdot \underline{\lambda}, G_\sigma^\vee}$ is nonempty. To summarize, we have the following sequence:

$$\mathbf{C}_{N\underline{\lambda},(G_\sigma)^\vee} \neq \emptyset \overset{\text{Theorem 3.25}}{\Longleftrightarrow} (\mathbf{C}_{N\underline{\lambda},G^\vee})^{\sigma^\vee} \neq \emptyset \implies \mathbf{C}_{N\underline{\lambda},G^\vee} \neq \emptyset \overset{\text{Assumption}}{\Longrightarrow} \mathbf{C}_{k\underline{\lambda},G^\vee} \neq \emptyset$$
$$\overset{\text{Theorem 3.29}}{\Longrightarrow} (\mathbf{C}_{kc_\sigma\cdot\underline{\lambda},G^\vee})^{\sigma^\vee} \neq \emptyset \overset{\text{Theorem 3.25}}{\Longleftrightarrow} \mathbf{C}_{kc_\sigma\cdot\underline{\lambda},(G_\sigma)^\vee} \neq \emptyset.$$

Theorem 1.2 is proved.

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