



Generalized semi-invariant distributions on p -adic spaces

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Abstract In this paper we investigate some methods on calculating the spaces of generalized semi-invariant distributions on p -adic spaces. Using homological methods, we give a criterion of automatic extension of (generalized) semi-invariant distributions. Based on the meromorphic continuations of Igusa zeta integrals, we give another criteria with purely algebraic geometric conditions, on the extension of generalized semi-invariant distributions.

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1 Introduction

Following Bernstein-Zelevinsky [6], we define an ℓ -space to be a topological space which is Hausdorff, locally compact, totally disconnected and secondly countable. An ℓ -group is a topological group whose underlying topological space is an ℓ -space. Let G be an ℓ -group acting continuously on an ℓ -space X . We may ask a general question about how to describe all semi-invariant distributions on X with respect to the action of G , that is, to determine the space

$$D(X)^\chi := \text{Hom}_G(\mathcal{S}(X), \chi) \quad (1)$$

for a fixed character $\chi : G \rightarrow \mathbb{C}^\times$ (all characters of ℓ -groups are assumed to be locally constant in this paper). Here $\mathcal{S}(X)$ denotes the space of Bruhat-Schwartz functions on X , namely, the space of compactly supported, locally constant complex valued functions on X . Here and as usual, when no confusion is possible, we do not distinguish a representation with its underlying (complex) vector space. In particular, we do not distinguish a character with the representation attached to it on the one-dimensional vector space \mathbb{C} . We call an element of (1) a χ -invariant distribution on X . Many problems on number theory and representation theory of p -adic groups end up to the problems on semi-invariant distributions of this kind. There are quite a lot of techniques on the vanishing of invariant distributions. It seems to us that the constructions of semi-invariant distributions are still not fully developed.

We suggest in this paper that, to describe all semi-invariant distributions on the ℓ -spaces, it would be more achievable to first consider some more general distributions. They are the generalized semi-invariant distributions as in the following definition.

Definition 1.1 Let V be a (non-necessary smooth) representation of G . A vector $v \in V$ is called a generalized invariant vector if there is a $k \in \mathbb{N}$ such that

$$(g_0 - 1)(g_1 - 1) \dots (g_k - 1).v = 0 \quad \text{for all } g_0, g_1, \dots, g_k \in G.$$

A generalized χ -invariant distribution on X is defined to be a generalized invariant vector in the representation $\text{Hom}_{\mathbb{C}}(\mathcal{S}(X), \chi)$ of G .

Here and as usual, the group G acts on $\text{Hom}_{\mathbb{C}}(\mathcal{S}(X), \chi)$ as in the Eq. (5) of Sect. 2.2. The set of non-negative integers is denoted by \mathbb{N} .

When X is a G -homogeneous space (to be more precise, this means that the action of G on X is transitive, and for every $x \in X$, the orbit map $G \rightarrow X, g \mapsto g.x$ is open), the space (1) is at most one-dimensional. We introduce the following definition.

Definition 1.2 When X is a homogeneous space of G , we say that X is χ -admissible if the space (1) is non-zero.

We are mainly concerned with ℓ -spaces and ℓ -groups of algebraic geometric origin. Throughout the paper, we fix a non-archimedean local field F of characteristic zero.

Definition 1.3 Assume that $G = \mathbf{G}(F)$ for some linear algebraic group \mathbf{G} defined over F . Let X be an algebraic variety over F , with an algebraic action of \mathbf{G} . We say that a G -orbit $O \subset X(F)$ is weakly χ -admissible if the homogeneous space $G/\mathbf{G}_x^\circ(F)$ is χ -admissible, where $x \in O$, and \mathbf{G}_x° denotes the identity connected component of the stabilizer \mathbf{G}_x of x in \mathbf{G} .

The above definition is certainly independent of the choice of $x \in O$. As usual, by an algebraic variety over F , we mean a scheme over F which is separated, reduced, and of finite type. A linear algebraic group over F is a group scheme over F which is an affine variety as a scheme.

The first main result we obtain in this paper is the following automatic extension theorem for semi-invariant distributions and generalized semi-invariant distributions.

Theorem 1.4 *Let \mathbf{G} be a linear algebraic group defined over F , acting algebraically on an algebraic variety X over F . Let χ be a character of $\mathbf{G}(F)$, and let U be a \mathbf{G} -stable open subvariety of X . Assume that every $\mathbf{G}(F)$ -orbit in $(X \setminus U)(F)$ is not weakly χ -admissible. Then every χ -invariant distribution on $U(F)$ uniquely extends to a χ -invariant distribution on $X(F)$, and every generalized χ -invariant distribution on $U(F)$ uniquely extends to a generalized χ -invariant distribution on $X(F)$.*

In Theorem 1.4, if we replace “weakly χ -admissible” by “ χ -admissible”, then the uniqueness assertion of the theorem remains true, by the localization principle of Bernstein-Zelevinsky [6, Theorem 6.9]. In particular it implies that if every $\mathbf{G}(F)$ -orbit in $X(F)$ is not χ -admissible, then there is no nonzero generalized χ -invariant distribution on $X(F)$. But the extendability may fail in general, as shown in the following example. Let $G = \{\pm 1\} \times F^\times$, which acts on $X := \{(x, y) \in F^2 \mid xy = 0\}$ by

$$(1, a).(x, y) := (ax, a^{-1}y) \quad \text{and} \quad (-1, a).(x, y) := (a^{-1}y, ax),$$

for all $a \in F^\times$ and $(x, y) \in X$. Let χ be the non-trivial quadratic character of G which is trivial of F^\times . Then the orbit $\{(0, 0)\}$ is weakly χ -admissible, but not χ -admissible. It is well known that a non-zero χ -invariant distribution on $X \setminus \{(0, 0)\}$ does not extends to a χ -invariant distribution on X .

The idea of generalized semi-invariant distributions can even be dated back to the famous Tate’s thesis. It has rooted in the dimension one property of the space of semi-invariant distributions on F with respect to the multiplicative action of F^\times . Let χ denote a character of F^\times for the moment. As a simple application of Theorem 1.4, we know that

$$\dim \text{Hom}_{F^\times}(\mathcal{S}(F), \chi) = 1 \tag{2}$$

when χ is non-trivial. However, Theorem 1.4 is no longer applicable when χ is trivial. Instead, when χ is trivial, we consider the meromorphic continuation of the zeta integral

$$\int_F \phi(x)|x|^s dx, \quad \phi \in \mathcal{S}(F).$$

This zeta integral has simple pole at $s = -1$. Taking all coefficients of the Laurent expansion of the zeta integral at $s = -1$, we actually get all generalized χ -invariant distributions on F . By considering the natural action of F^\times on this space of all generalized χ -invariant distributions, one concludes that (2) also holds when χ is trivial.

A key observation of the above argument is that generalized invariant distributions on F^\times extends to generalized invariant distributions on F . The second main result of this paper is the following generalization of this observation.

Theorem 1.5 *Let \mathbf{G} be a linear algebraic group over F . Let \mathbf{X} be an algebraic variety over F so that \mathbf{G} acts algebraically on it with an open orbit $\mathbf{U} \subset \mathbf{X}$. Assume that there is a semi-invariant regular function f on \mathbf{X} , with the following properties:*

- *f does not vanish on \mathbf{U} , and $\mathbf{X}_f \setminus \mathbf{U}$ has codimension ≥ 2 in \mathbf{X}_f , where \mathbf{X}_f denotes the complement in \mathbf{X} of the zero locus of f ;*
- *the variety \mathbf{X}_f has Gorenstein rational singularities.*

Let χ be a character of $\mathbf{G}(F)$ which is trivial on $\mathbf{N}(F)$, where \mathbf{N} denotes the unipotent radical of \mathbf{G} . Then every generalized χ -invariant distribution on $\mathbf{U}(F)$ extends to a generalized χ -invariant distribution on $\mathbf{X}(F)$.

Here a regular function f on \mathbf{X} being semi-invariant means that, there exists an algebraic character ν of \mathbf{G} over F such that

$$f(g.x) = \nu(g)f(x), \quad \text{for all } g \in \mathbf{G}(\bar{F}) \text{ and } x \in \mathbf{X}(\bar{F}), \tag{3}$$

where \bar{F} denotes an algebraic closure of F .

- Remark* (a) Let \mathbf{G} be a linear algebraic group over F , acting algebraically on an algebraic variety Y over F . We say that Y is \mathbf{G} -homogeneous, or Y is a \mathbf{G} -homogeneous space, if the action of $\mathbf{G}(\bar{F})$ on $Y(\bar{F})$ is transitive. In general, a subvariety Z of Y is called a \mathbf{G} -orbit if it is \mathbf{G} -stable and \mathbf{G} -homogeneous.
- (b) We say that a subvariety Z of an algebraic variety Y has codimension $\geq r$ ($r \in \mathbb{N}$) if

$$r + \dim_x Z \leq \dim_x Y \quad \text{for all } x \in Z.$$

- (c) The notion of Gorenstein rational singularity is reviewed in Sect. 6.7.
- (d) A variant of Theorem 1.5 is stated in Theorem 6.34, where X_f is only required to have rational singularities, but we additionally assume that there exists a nonzero semi-invariant algebraic volume form on U .

In order to prove Theorem 1.5, as in the case of Tate's thesis we need to employ the theory of zeta integrals. For each generalized χ -invariant distribution μ on $\mathbf{U}(F)$, it turns out that μ is a definable measure (Definition 5.9) and it is locally finite on $X_f(F)$ (Theorem 6.31). We attach a zeta integral

$$Z_{\mu,f}(\phi, s) := \int_{X_f(F)} \phi(x) |f(x)|^s d\mu(x),$$

for every $\phi \in \mathcal{S}(X(F))$. The meromorphic continuation of $Z_{\mu,f}$ is a consequence of a general fact of Igusa zeta integrals on semi-algebraic spaces, which is proved in Theorem 5.13.

The structure of the paper is as follows. In Sect. 2, we introduce the basics of generalized homomorphisms and generalized extensions, and we prove a vanishing theorem of generalized extensions (Theorem 2.12). In Sect. 3, we prove a localization principle for extensions in the settings of equivariant ℓ -sheaves (Theorem 3.2). Section 4 is devoted to a proof of our first main theorem. We first establish the generalized version of Frobenius reciprocity and Shapiro Lemma. Then by results in Sects. 2 and 3, we prove a higher version of automatic extension theorem (Theorem 4.10), which contains Theorem 1.4 as a special case.

In Sect. 5, we introduce p -adic semi-algebraic spaces and the measure theory on them. We prove the meromorphic continuation of Igusa zeta integral on general semi-algebraic spaces (Theorem 5.13) after the works of Denef, Cluckers et al. In Sect. 6, we prove the second main theorem as follows. We first prove that any generalized semi-invariant distribution on algebraic homogeneous spaces is a definable measure (Theorem 6.16, Proposition 6.22), in the sense of Definition 5.9. Then we prove that it is locally finite (Theorem 6.31) if the boundary has Gorenstein rational singularities. In the end Theorem 1.5 follows from Theorem 5.13.

As an illustration, we determine all generalized semi-invariant distributions on matrix spaces in Sect. 7. In this example we employ intensively the automatic extension theorem and meromorphic continuations of distributions.

2 Generalized homomorphisms and generalized extensions

2.1 The space of generalized invariant vectors

Let G be an ℓ -group as in the Introduction. By a representation of G , we mean a complex vector space together with a linear action of G on it. A vector in a representation of G is said to be smooth if it is fixed by an open subgroup of G . A representation of G is said to be smooth if all its vectors are smooth.

Let V be a representations of G . Define a sequence

$$V^{G,0} \subset V^{G,1} \subset V^{G,2} \subset \dots$$

of subrepresentations of V by

$$V^{G,k} := \{v \in V \mid (g_0 - 1)(g_1 - 1) \cdots (g_k - 1).v = 0 \text{ for all } g_0, g_1, \dots, g_k \in G\}. \quad (4)$$

Put

$$V^{G,\infty} := \bigcup_{k \in \mathbb{N}} V^{G,k}.$$

A vector of $V^{G,\infty}$ is called a generalized G -invariant vector in V .

Definition 2.1 A representation of G is said to be locally unipotent if it is smooth and all its vectors are generalized G -invariant.

At least when V is a smooth representation, it is elementary to see that every compact subgroup of G acts trivially on $V^{G,\infty}$. Define G° to be the subgroup of G generated by all compact subgroups of G , which is an open normal subgroup of G (similar notation will be used without further explanation for other ℓ -groups). Put

$$\Lambda_G := G/G^\circ.$$

Then when V is smooth, $V^{G,\infty}$ descends to a locally unipotent representation of Λ_G .

Recall from the Introduction that F is a non-archimedean local field of characteristic zero.

Proposition 2.2 (see [3, Chapter II, Proposition 22]) *Assume that $G = \mathbf{G}(F)$ for some connected linear algebraic group \mathbf{G} defined over F . Then Λ_G is a free abelian group whose rank equals the dimensional of the maximal central split torus of a Levi component of \mathbf{G} .*

2.2 Generalized homomorphisms

Let V_1, V_2 be two smooth representations of G . Then $\text{Hom}_{\mathbb{C}}(V_1, V_2)$ is naturally a representation of G :

$$g \cdot \phi(v) := g \cdot (\phi(g^{-1} \cdot v)), \quad \phi \in \text{Hom}_{\mathbb{C}}(V_1, V_2), \quad v \in V_1. \tag{5}$$

For each $k = 0, 1, 2, \dots, \infty$, put

$$\text{Hom}_{G,k}(V_1, V_2) := (\text{Hom}_{\mathbb{C}}(V_1, V_2))^{G,k}.$$

We call a vector in $\text{Hom}_{G,\infty}(V_1, V_2)$ a generalized homomorphism from V_1 to V_2 .

Lemma 2.3 *For each open compact subgroup K of G , one has that*

$$\text{Hom}_{G,\infty}(V_1, V_2) \subset \text{Hom}_K(V_1, V_2).$$

In particular, every generalized homomorphism from V_1 to V_2 is a smooth vector of $\text{Hom}_{\mathbb{C}}(V_1, V_2)$.

Proof Write

$$V_1 = \bigoplus_{i \in I} V_{1,i}$$

as a direct sum of finite dimensional representations of K . Then one has that

$$\begin{aligned} & \text{Hom}_{G,\infty}(V_1, V_2) \\ & \subset \text{Hom}_{K,\infty}(V_1, V_2) \\ & \subset \prod_{i \in I} \text{Hom}_{K,\infty}(V_{1,i}, V_2) \\ & = \prod_{i \in I} \text{Hom}_K(V_{1,i}, V_2) \quad (\text{since } \text{Hom}_{\mathbb{C}}(V_{1,i}, V_2) \text{ is a smooth representation of } K) \\ & = \text{Hom}_K(V_1, V_2). \end{aligned}$$

□

By Lemma 2.3, we know that $\text{Hom}_{G,k}(V_1, V_2)$ is a locally unipotent representation of Λ_G ($k = 0, 1, 2, \dots, \infty$). The following lemma is obvious.

Lemma 2.4 *One has that*

$$\text{Hom}_{G,\infty}(V_1, V_2) = 0 \quad \text{if and only if} \quad \text{Hom}_G(V_1, V_2) = 0.$$

The following lemma is routine to check. We omit the details.

Lemma 2.5 (a) *Let V_1, V_2, V_3 be smooth representations of G . Let $k_1, k_2 \in \{0, 1, 2, \dots, \infty\}$. Then*

$$\phi_2 \circ \phi_1 \in \text{Hom}_{G,k_1+k_2}(V_1, V_3)$$

for all $\phi_1 \in \text{Hom}_{G,k_1}(V_1, V_2)$ and $\phi_2 \in \text{Hom}_{G,k_2}(V_2, V_3)$.

(b) Let V_1, V_2, V'_1, V'_2 be smooth representations of G . Let $k_1, k_2 \in \{0, 1, 2, \dots, \infty\}$. Then

$$\phi_1 \otimes \phi_2 \in \text{Hom}_{G, k_1+k_2}(V_1 \otimes V_2, V'_1 \otimes V'_2)$$

for all $\phi_1 \in \text{Hom}_{G, k_1}(V_1, V'_1)$ and $\phi_2 \in \text{Hom}_{G, k_2}(V_2, V'_2)$.

2.3 Generalized homomorphisms and homomorphisms

Denote by $\mathbb{C}[\Lambda_G]$ the group algebra of Λ_G . Denote by I_G the augmentation ideal of $\mathbb{C}[\Lambda_G]$, namely,

$$I_G := \left\{ \sum_{g \in \Lambda_G} a_g g \in \mathbb{C}[\Lambda_G] \mid \sum_{g \in \Lambda_G} a_g = 0 \right\}.$$

For each $k \in \mathbb{N}$, put

$$J_{G,k} := \mathbb{C}[\Lambda_G]/(I_G)^{k+1}.$$

We view it as a locally unipotent representation of G through left translations. The following lemma is routine to check.

Lemma 2.6 *Let $k \in \mathbb{N}$. For each smooth representation V of G , the map*

$$\text{Hom}_G(J_{G,k}, V) \rightarrow V^{G,k}, \quad \phi \mapsto \phi(1)$$

is a well-defined isomorphism of locally unipotent representations of G . Here $\text{Hom}_G(J_{G,k}, V)$ is viewed as a smooth representation of G by

$$(g \cdot \phi)(x) := \phi(x\bar{g}), \quad g \in G, \quad \phi \in \text{Hom}_G(J_{G,k}, V), \quad x \in J_{G,k},$$

where \bar{g} denotes the image of g under the natural map $G \rightarrow J_{G,k}$.

More generally, we have the following lemma.

Lemma 2.7 *Let $k \in \mathbb{N}$. For all smooth representations V_1 and V_2 of G , the map*

$$\text{Hom}_G(J_{G,k} \otimes V_1, V_2) \rightarrow \text{Hom}_{G,k}(V_1, V_2), \quad \phi \mapsto \phi|_{V_1}$$

is a well-defined isomorphism of locally unipotent representations of G . Here V_1 is identified with the subspace $1 \otimes V_1$ of $J_{G,k} \otimes V_1$, and $\text{Hom}_G(J_{G,k} \otimes V_1, V_2)$ is viewed as a smooth representation of G by

$$(g \cdot \phi)(x \otimes v) := \phi(x\bar{g} \otimes v), \quad g \in G, \quad \phi \in \text{Hom}_G(J_{G,k}, V), \quad x \in J_{G,k}, \quad v \in V_1,$$

where \bar{g} denotes the image of g under the natural map $G \rightarrow J_{G,k}$.

Proof We have the G -equivariant identifications

$$\begin{aligned} \text{Hom}_{G,k}(V_1, V_2) &= \text{Hom}_{\mathbb{C}}(V_1, V_2)^{G,k} \\ &= \text{Hom}_G(\mathbb{J}_{G,k}, \text{Hom}_{\mathbb{C}}(V_1, V_2)) \\ &= \text{Hom}_G(\mathbb{J}_{G,k} \otimes V_1, V_2). \end{aligned}$$

Therefore the lemma follows. □

Lemma 2.6 implies that

$$V^{G,\infty} = \varinjlim_k \text{Hom}_G(\mathbb{J}_{G,k}, V),$$

for all smooth representations V of G . Likewise, Lemma 2.7 implies that

$$\text{Hom}_{G,\infty}(V_1, V_2) = \varinjlim_k \text{Hom}_G(\mathbb{J}_{G,k} \otimes V_1, V_2),$$

for all smooth representations V_1 and V_2 of G .

2.4 Schwartz inductions

We briefly recall the Schwartz inductions in this subsection. Let H be a closed subgroup of G . Let V_0 be a smooth representation of H . Define the un-normalized Schwartz induction $\text{ind}_H^G V_0$ to be the space of all V_0 -valued locally constant functions ϕ on G such that

- $\phi(hg) = h.\phi(g)$, for all $h \in H, g \in G$; and
- ϕ has compact support modulo (the left translations of) H .

It is a smooth representation of G under right translations. The following lemma is well known and easy to check.

Lemma 2.8 *Let V be a smooth representation of G . Then the linear map*

$$\begin{aligned} V \otimes \text{ind}_H^G V_0 &\rightarrow \text{ind}_H^G (V|_H \otimes V_0), \\ v \otimes \phi &\mapsto (g \mapsto g.v \otimes \phi(g)) \end{aligned} \tag{6}$$

is a well defined isomorphism of smooth representations of G .

2.5 Generalized extensions

Denote by $\mathcal{M}(G)$ the category of smooth representations of G (the morphisms of this category are G -intertwining linear maps). By a projective smooth representation of G , we mean a projective object of the category $\mathcal{M}(G)$.

Lemma 2.9 *Let V_1, V_2 be two smooth representations of G . If V_1 or V_2 is projective, then $V_1 \otimes V_2$ is projective.*

Proof This is well known. We sketch a proof for the convenience of the reader. Without loss of generality, assume that V_2 is projective. Note that V_2 is isomorphic to a quotient of $\text{ind}_{\{1\}}^G V_2$. Since it is projective, it is isomorphic to a direct summand of $\text{ind}_{\{1\}}^G V_2$. Therefore $V_1 \otimes V_2$ is isomorphic to a direct summand of

$$V_1 \otimes \left(\text{ind}_{\{1\}}^G V_2\right) \cong \text{ind}_{\{1\}}^G(V_1 \otimes V_2) \quad (\text{by Lemma 2.7}).$$

By [7, Theorem A.4], $\text{ind}_{\{1\}}^G(V_1 \otimes V_2)$ is projective. Therefore $V_1 \otimes V_2$ is also projective. □

Lemma 2.10 *Let V_1 and V_2 be two smooth representations of G . Let $P_\bullet \rightarrow V_1$ be a projective resolution of V_1 , and let $V_2 \rightarrow I^\bullet$ be an injective resolution of V_2 . Then for each $i \in \mathbb{Z}$, the i -th cohomology of the complex $\text{Hom}_{G,k}(P_\bullet, V_2)$ and the i -th cohomology of the complex $\text{Hom}_{G,k}(V_1, I^\bullet)$ are both canonically isomorphic to*

$$\begin{cases} \text{Ext}_G^i(J_{G,k} \otimes V_1, V_2), & \text{for } k \in \mathbb{N}; \\ \varinjlim_r \text{Ext}_G^i(J_{G,r} \otimes V_1, V_2), & \text{for } k = \infty. \end{cases}$$

Proof First we assume that $k \in \mathbb{N}$. Then

$$\text{Hom}_{G,k}(P_\bullet, V_2) = \text{Hom}_G(J_{G,k} \otimes P_\bullet, V_2), \quad (\text{by Lemma 2.6}). \tag{7}$$

By Lemma 2.9, $J_{G,k} \otimes P_\bullet \rightarrow J_{G,k} \otimes V_1$ is a projective resolution of $J_{G,k} \otimes V_1$. Therefore the i -th cohomology of the complex (7) is canonically isomorphic to $\text{Ext}_G^i(J_{G,k} \otimes V_1, V_2)$. On the other hand, it is obvious that the i -th cohomology of the complex

$$\text{Hom}_{G,k}(V_1, I^\bullet) = \text{Hom}_{G,k}(J_{G,k} \otimes V_1, I^\bullet) \tag{8}$$

is canonically isomorphic to $\text{Ext}_G^i(J_{G,k} \otimes V_1, V_2)$.

The Lemma for $k = \infty$ then follows since taking cohomology commutes with taking direct limits. □

Denote by $\mathcal{M}_u(\Lambda_G)$ the category of all locally unipotent representations of Λ_G . For each $k = 0, 1, 2, \dots, \infty$, we have a bi-functor

$$\text{Hom}_{G,k}(\cdot, \cdot) : \mathcal{M}(G)^{\text{op}} \times \mathcal{M}(G) \rightarrow \mathcal{M}_u(\Lambda_G).$$

In view of Lemma 2.10, write

$$\text{Ext}_{G,k}^i(\cdot, \cdot) : \mathcal{M}(G)^{\text{op}} \times \mathcal{M}(G) \rightarrow \mathcal{M}_u(\Lambda_G)$$

for its i -th left derived bi-functor ($i \in \mathbb{Z}$).

Let Γ be a directed set, i.e. Γ is a partially ordered set with a partial order \leq and for any $\gamma, \gamma' \in \Gamma$, there exists $\gamma'' \in \Gamma$, such that $\gamma \leq \gamma''$ and $\gamma' \leq \gamma''$. We can view Γ as a category where morphisms come from the partial order. Let Γ^{op} be the opposite category of Γ . Let \mathcal{C} be an abelian category. A directed (resp. directed

inverse) system of objects in \mathcal{C} is a functor from Γ (resp. Γ^0) to \mathcal{C} . We can write such a system as $\{V_\gamma\}_{\gamma \in \Gamma}$, where $V_\gamma \in \mathcal{C}$ and for any $\gamma \leq \gamma'$ in Γ we associate a morphism $\phi_{\gamma\gamma'} : V_\gamma \rightarrow V_{\gamma'}$ (resp. $\phi_{\gamma\gamma'} : V_{\gamma'} \rightarrow V_\gamma$). We call a directed (directed inverse) system $\{V_\gamma\}_{\gamma \in \Gamma}$ injective (resp. surjective) if for any $\gamma \leq \gamma'$ the morphism $\phi_{\gamma\gamma'}$ is injective (resp. surjective).

Lemma 2.11 *Let V be a smooth representation of G , and let $\{V_\gamma\}_{\gamma \in \Gamma}$ be an injective directed system of smooth representations of G where Γ is a countable directed set. Let $k \in \mathbb{N}$. If for all $i \in \mathbb{Z}$ and $\gamma \in \Gamma$, $\text{Ext}_{G,k}^i(V_\gamma, V) = 0$, then $\text{Ext}_{G,k}^i(\varinjlim_\gamma V_\gamma, V) = 0$ for all $i \in \mathbb{Z}$.*

Proof In view of Lemma 2.10, $\text{Ext}_{G,k}^i(\varinjlim_\gamma V_\gamma, V)$ can be computed as i -th cohomology of $\text{Hom}_{G,k}(\varinjlim_\gamma V_\gamma, I^\bullet)$, where $I^\bullet = \{\dots \rightarrow 0 \rightarrow I^0 \rightarrow I^1 \rightarrow \dots\}$ is an injective resolution of V . We have the following isomorphisms,

$$\begin{aligned} \text{Hom}_{G,k}(\varinjlim_\gamma V_\gamma, I^\bullet) &\simeq \text{Hom}_G(\mathbb{J}_{G,k} \otimes (\varinjlim_\gamma V_\gamma), I^\bullet) \\ &\simeq \varprojlim_\gamma \text{Hom}_G(\mathbb{J}_{G,k} \otimes V_\gamma, I^\bullet) \\ &\simeq \varprojlim_\gamma \text{Hom}_{G,k}(V_\gamma, I^\bullet), \end{aligned}$$

where the first and the third isomorphisms follow from Lemma 2.7, and the second isomorphism is a general property of Hom functor. Therefore it suffices to show that the inverse limit of the system of complexes $\{\text{Hom}_{G,k}(V_\gamma, I^\bullet)\}_{\gamma \in \Gamma}$ is acyclic.

Let $X_\gamma^\bullet = \{\dots \rightarrow 0 \rightarrow X_\gamma^0 \rightarrow X_\gamma^1 \rightarrow \dots\}$ be the cochain complex $\text{Hom}_{G,k}(V_\gamma, I^\bullet)$. We get a directed inverse system of cochain complexes $\{X_\gamma^\bullet\}_{\gamma \in \Gamma}$. The directed inverse system $\{X_\gamma^i\}_{\gamma \in \Gamma}$ is surjective for each i since I^i is an injective module and $\phi_{\gamma\gamma'} : V_\gamma \rightarrow V_{\gamma'}$ is an injective morphism. By assumption on the vanishing of $\text{Ext}_{G,k}^i(V_\gamma, V)$ for any i and γ , we get an acyclic complex of surjective directed inverse systems,

$$\dots \rightarrow 0 \rightarrow \{X_\gamma^0\}_{\gamma \in \Gamma} \xrightarrow{\{d_\gamma^0\}} \{X_\gamma^1\}_{\gamma \in \Gamma} \xrightarrow{\{d_\gamma^1\}} \dots$$

Let Ker_γ^i be the kernel of d_γ^i , and let Im_γ^i be the image of d_γ^i . For every i , we have $\text{Ker}_\gamma^i = \text{Im}_\gamma^{i-1}$. Note that $\text{Ker}_\gamma^1 = X_\gamma^0$ and we have short exact sequences

$$0 \rightarrow \text{Ker}_\gamma^i \rightarrow X_\gamma^i \rightarrow \text{Ker}_\gamma^{i+1} \rightarrow 0.$$

By induction it is easy to see that for all i the directed inverse system $\{\text{Ker}_\gamma^i\}_{\gamma \in \Gamma}$ is surjective. Hence for all i we have the following short exact sequences (see [12, Lemma 10.85.4])

$$0 \rightarrow \varprojlim_\gamma \text{Ker}_\gamma^i \rightarrow \varprojlim_\gamma X_\gamma^i \rightarrow \varprojlim_\gamma \text{Ker}_\gamma^{i+1} \rightarrow 0.$$

Combining all these short exact sequences, we conclude that the complex $\lim_{\leftarrow \gamma} X_\gamma^\bullet$ is acyclic. □

2.6 A vanishing theorem of generalized extensions

The main result of this subsection is the following theorem.

Theorem 2.12 *Assume that $G = \mathbf{G}(F)$ for some connected linear algebraic group \mathbf{G} defined over F . Let V_1 and V_2 be two smooth representations of G . Assume that there are two distinct characters χ_1 and χ_2 of G such that both $V_1 \otimes \chi_1^{-1}$ and $V_2 \otimes \chi_2^{-1}$ are locally unipotent as representations of G , then*

$$\text{Ext}_{G,k}^i(V_1, V_2) = 0, \quad i \in \mathbb{Z}, k = 0, 1, 2, \dots, \infty.$$

We remark that Theorem 2.12 fails without the connectedness assumption on \mathbf{G} . Instead, we will use the following corollary in the disconnected case.

Corollary 2.13 *Let G be an ℓ -group which contains $\mathbf{G}(F)$ as an open normal subgroup of finite index, where \mathbf{G} is a connected linear algebraic group defined over F . Let V_1 and V_2 be two smooth representations of G . Assume that there are two distinct characters χ_1 and χ_2 of $\mathbf{G}(F)$ such that both $(V_1)|_{\mathbf{G}(F)} \otimes \chi_1^{-1}$ and $(V_2)|_{\mathbf{G}(F)} \otimes \chi_2^{-1}$ are locally unipotent as representations of $\mathbf{G}(F)$, then*

$$\text{Ext}_{G,k}^i(V_1, V_2) = 0, \quad i \in \mathbb{Z}, k = 0, 1, 2, \dots, \infty.$$

Proof Note that the tensor product of two locally unipotent representations is also a locally unipotent representation. By Lemma 2.10, it suffices to prove the corollary for $k = 0$. Let P_\bullet be a projective resolution of V_1 . Then $(P_\bullet)|_{\mathbf{G}(F)}$ is a projective resolution of $(V_1)|_{\mathbf{G}(F)}$. By Theorem 2.12, the complex $\text{Hom}_{\mathbf{G}(F)}(P_\bullet, V_2)$ is acyclic. Therefore the complex $\text{Hom}_G(P_\bullet, V_2)$, which equals the complex $(\text{Hom}_{\mathbf{G}(F)}(P_\bullet, V_2))^{G/\mathbf{G}(F)}$ of the $G/\mathbf{G}(F)$ -invariant vectors, is also acyclic. This proves the corollary. □

The rest of this subsection is devoted to a proof of Theorem 2.12.

Lemma 2.14 *Let V_1 and V_2 be two smooth representations of an ℓ -group G . Then for each character χ of G , there is an isomorphism*

$$\text{Ext}_{G,k}^i(V_1, V_2) \cong \text{Ext}_{G,k}^i(V_1 \otimes \chi, V_2 \otimes \chi), \quad i \in \mathbb{Z}, k = 0, 1, 2, \dots, \infty$$

of locally unipotent representations of Λ_G .

Proof Take an injective resolution

$$0 \rightarrow V_2 \rightarrow I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \dots$$

of V_2 . Then

$$0 \rightarrow V_2 \otimes \chi \rightarrow I_0 \otimes \chi \rightarrow I_1 \otimes \chi \rightarrow I_2 \otimes \chi \rightarrow \dots$$

is an injective resolution of $V_2 \otimes \chi$. Therefore the lemma follows. □

The following Lemma is well known and is an easy consequence of Lemma 2.9.

Lemma 2.15 *Let V_1, V_2 be two smooth representations of an ℓ -group G . Then for all $i \in \mathbb{Z}$,*

$$\text{Ext}_G^i(V_1, V_2^\vee) \cong H_i(G, V_1 \otimes V_2)^*.$$

In particular,

$$H_i(G, V)^* \cong \text{Ext}_G^i(V, \mathbb{C}),$$

for all smooth representation V of G .

Here and henceforth, a superscript “ \vee ” indicates the smooth contragredient of a smooth representation, a superscript “ $*$ ” indicates the space of all linear functionals, and “ H_i ” indicates the i -th homology group.

Lemma 2.16 *Let U be a unipotent linear algebraic group over F , and put $U := U(F)$. Let χ be a character of U . Then for each $i \in \mathbb{Z}$,*

$$H_i(U, \chi) = 0 \quad \text{if } i \neq 0 \text{ or } \chi \text{ is non-trivial.}$$

Proof By [3, Proposition 10, Section 3.3], the coinvariant functor $V \mapsto V_U$ from the category $\mathcal{M}(U)$ to the category of complex vector spaces is exact. This implies the lemma. □

Similar to Lemma 2.16, we have the following lemma for semisimple groups.

Lemma 2.17 *Let S be a connected semisimple linear algebraic group over F , and put $S := S(F)$. Let χ be a character of S . Then for each $i \in \mathbb{Z}$,*

$$H_i(S, \chi) = 0 \quad \text{if } i \neq 0 \text{ or } \chi \text{ is non-trivial.}$$

Proof In view of Lemma 2.15, this is implied by [7, Theorem A.13]. □

The following lemma is a variant of Hochschild-Serre spectral sequence, see [7, Proposition A.9].

Lemma 2.18 *Let H be a closed normal subgroup of an ℓ -group G . Let V and W be smooth representations of G , with H acting trivially on W . Then there is a spectral sequence*

$$E_2^{p,q} = \text{Ext}_{G/H}^p(H_q(H, V), W) \Rightarrow \text{Ext}_G^{p+q}(V, W).$$

Generalizing Lemma 2.17, we have the following lemma for reductive groups.

Lemma 2.19 *Let L be a connected reductive linear algebraic group over F , and put $L := L(F)$. Let χ be a character of L° . Then for each $i \in \mathbb{Z}$,*

$$H_i(L^\circ, \chi) = 0 \quad \text{if } i \neq 0 \text{ or } \chi \text{ is non-trivial.}$$

Proof Write S for the derived subgroup of L , and put $S := S(F)$. Lemma 2.15 and Lemma 2.18 imply that

$$H_i(L^\circ, \chi)^* \cong \text{Ext}_{L^\circ/S}^i(H_0(S, \chi), \mathbb{C}). \tag{9}$$

If $i \neq 0$, then the right hand side of (9) vanishes since L°/S is compact. The lemma is obvious for $i = 0$. □

Lemma 2.20 *Let (X, \mathcal{O}_X) be a ringed space. Let \mathcal{F}_1 and \mathcal{F}_2 be two \mathcal{O}_X -modules. If the supports of \mathcal{F}_1 and \mathcal{F}_2 are disjoint, then*

$$\text{Ext}_{\mathcal{O}_X}^i(\mathcal{F}_1, \mathcal{F}_2) = 0, \quad i \in \mathbb{Z}.$$

Proof By the construction of injective resolutions as in [15, Chapter III, Proposition 2.2], we know that there is an injective resolution $\mathcal{F}_2 \rightarrow \mathcal{I}_\bullet$ such that the support of \mathcal{I}_i is contained in that of \mathcal{F}_2 ($i \in \mathbb{Z}$). Therefore the lemma follows. □

Lemma 2.21 *Let Λ be a finitely generated free abelian group. Let V_1 and V_2 be two representations of Λ . Assume that there are two distinct characters χ_1 and χ_2 of Λ such that both $V_1 \otimes \chi_1^{-1}$ and $V_2 \otimes \chi_2^{-1}$ are locally unipotent as representations of Λ , then*

$$\text{Ext}_\Lambda^i(V_1, V_2) = 0, \quad i \in \mathbb{Z}.$$

Proof Write $\mathbb{C}[\Lambda]$ for the complex group algebra attached to Λ . Then both V_1 and V_2 are $\mathbb{C}[\Lambda]$ -modules, and we have that

$$\text{Ext}_\Lambda^i(V_1, V_2) = \text{Ext}_{\mathbb{C}[\Lambda]}^i(V_1, V_2), \quad i \in \mathbb{Z}.$$

Denote by $\widetilde{\mathbb{C}[\Lambda]}$ the structure sheaf of the scheme $\text{Spec}(\mathbb{C}[\Lambda])$, and denote by \widetilde{V}_1 and \widetilde{V}_2 the quasi-coherent $\widetilde{\mathbb{C}[\Lambda]}$ -modules attached to V_1 and V_2 , respectively. Note that there exists a filtration $0 = V_1^0 \subset V_1^1 \subset V_1^2 \subset \dots$ of the representation V_1 of Λ such that $\bigcup_{k \geq 1} V_1^k = V_1$ and V_1^k/V_1^{k-1} is a direct sum of copies of χ_1 for every $k \geq 1$.

Using Lemma 2.11, we are reduced to show that for all k

$$\text{Ext}_\Lambda^i(V_1^k, V_2) = 0, \quad i \in \mathbb{Z}.$$

For any vector space W , we always have

$$\text{Ext}_\Lambda^i(W \otimes \chi_1, V_2) = W^* \otimes \text{Ext}_\Lambda^i(\chi_1, V_2)$$

for every i , where W^* is the dual vector space of W . By induction on k , we assume without loss of generality that $V_1 = \chi_1$. Then we have that (see [15, Chapter III, exercise 6.7])

$$\text{Ext}_{\mathbb{C}[\Lambda]}^i(V_1, V_2) = \text{Ext}_{\mathbb{C}[\Lambda]}^i(\widetilde{V}_1, \widetilde{V}_2), \quad i \in \mathbb{Z}.$$

Since $\chi_1 \neq \chi_2$, the supports of \widetilde{V}_1 and \widetilde{V}_2 are disjoint. Therefore the lemma follows by Lemma 2.20. \square

Lemma 2.22 *Let \mathbf{L} be a connected reductive linear algebraic group over F , and put $L := \mathbf{L}(F)$. Let χ be a non-trivial character of L , and let V be a locally unipotent representation of L . Then*

$$\text{Ext}_L^i(\chi, V) = 0, \quad \text{for all } i \in \mathbb{Z}.$$

Proof Note that L° acts trivially on V . Lemmas 2.18 and 2.19 imply that

$$\text{Ext}_L^i(\chi, V) \cong \text{Ext}_{L/L^\circ}^i(\mathbf{H}_0(L^\circ, \chi), V), \quad \text{for all } i \in \mathbb{Z}.$$

If $\chi|_{L^\circ}$ is non-trivial, then the above space vanishes. Now assume that $\chi|_{L^\circ}$ is trivial. Then $\mathbf{H}_0(L^\circ, \chi)$ is a non-trivial one-dimensional representation of L/L° . The lemma then follows by Lemma 2.21. \square

Now we come to the proof of Theorem 2.12. Using Lemma 2.14, we assume without loss of generality that χ_2 is trivial. Then χ_1 is non-trivial. As in the proof of Corollary 2.13, it suffices to prove Theorem 2.12 for $k = 0$. As in the proof of Lemma 2.21, we may use Lemma 2.11 to further assume that $V_1 = \chi_1$. Then what we need to prove is that

$$\text{Ext}_G^i(\chi_1, V_2) = 0, \quad i \in \mathbb{Z} \tag{10}$$

for all non-trivial character χ_1 of G , and all locally unipotent representation V_2 of G .

Denote by \mathbf{N} the unipotent radical of \mathbf{G} , and put $N := \mathbf{N}(F)$. Note that N acts trivially on V_2 since $N \subset G^\circ$. If χ_1 is non-trivial on N , then Lemma 2.18 and Lemma 2.16 imply that (10) holds. Now assume that χ_1 is trivial on N . Then Lemma 2.18 and Lemma 2.16 imply that

$$\text{Ext}_G^i(\chi_1, V_2) \cong \text{Ext}_{G/N}^i(\chi_1, V_2), \quad i \in \mathbb{Z},$$

which vanishes by Lemma 2.22. This finishes the proof of Theorem 2.12.

3 A localization principle for extensions

3.1 Equivariant ℓ -sheaves and the localization principle

Let X be an ℓ -space. We define an ℓ -sheaf on X to be a sheaf of complex vector spaces on X . For any ℓ -sheaf \mathcal{F} on X , let $\Gamma_c(\mathcal{F})$ denote the space of all global sections of

\mathcal{F} with compact support. In particular, $\mathcal{S}(X) = \Gamma_c(\mathbb{C}_X)$, where \mathbb{C}_X denotes the sheaf of locally constant \mathbb{C} -valued functions on X . For each $x \in X$, denote by \mathcal{F}_x the stalk of \mathcal{F} at x ; and for each $s \in \Gamma_c(\mathcal{F})$, denote by $s_x \in \mathcal{F}_x$ the germ of s at x . The set $\bigsqcup_{x \in X} \mathcal{F}_x$ carries a unique topology such that for all $s \in \Gamma_c(\mathcal{F})$, the map

$$X \rightarrow \bigsqcup_{x \in X} \mathcal{F}_x, \quad x \mapsto s_x$$

is an open embedding. Then $\Gamma_c(\mathcal{F})$ is naturally identified with the space of all compactly supported continuous sections of the map $\bigsqcup_{x \in X} \mathcal{F}_x \rightarrow X$.

Let G be an ℓ -group which acts continuously on an ℓ -space X .

Definition 3.1 (cf. [6, Section 1.17]) A G -equivariant ℓ -sheaf on X is an ℓ -sheaf \mathcal{F} on X , together with a continuous group action

$$G \times \bigsqcup_{x \in X} \mathcal{F}_x \rightarrow \bigsqcup_{x \in X} \mathcal{F}_x$$

such that for all $x \in X$, the action of each $g \in G$ restricts to a linear map $\mathcal{F}_x \rightarrow \mathcal{F}_{g.x}$.

Given a G -equivariant ℓ -sheaf \mathcal{F} on X , the space $\Gamma_c(\mathcal{F})$ is a smooth representation of G so that

$$(g.s)_{g.x} = g.s_x \quad \text{for all } g \in G, \quad x \in X, \quad s \in \Gamma_c(\mathcal{F}).$$

For each G -stable locally closed subset Z of X , the restriction $\mathcal{F}|_Z$ is clearly a G -equivariant ℓ -sheaf on Z .

The main purpose of this section is to prove the following localization principle for extensions.

Theorem 3.2 *Let \mathcal{F} be a G -equivariant ℓ -sheaf on X . Let Y be an ℓ -space with a continuous map $\pi : X \rightarrow Y$ so that $\pi(g.x) = \pi(x)$, for all $x \in X$ and $g \in G$. Let V_1, V_2 be two smooth representations of G , and let $i \in \mathbb{Z}$. Assume that*

$$\text{Ext}_G^i(\Gamma_c(\mathcal{F}|_{X_y}) \otimes V_1, V_2^\vee) = 0 \quad \text{for all } y \in Y,$$

where $X_y := \pi^{-1}(y)$. Then

$$\text{Ext}_G^i(\Gamma_c(\mathcal{F}) \otimes V_1, V_2^\vee) = 0.$$

By Lemma 2.10, Theorem 3.2 has the following obvious consequence.

Corollary 3.3 *Let \mathcal{F} and $\pi : X \rightarrow Y$ be as in Theorem 3.2. Let χ be a character of G . Let $k \in \mathbb{N}$, and let $i \in \mathbb{Z}$. Assume that*

$$\text{Ext}_{G,k}^i(\Gamma_c(\mathcal{F}|_{X_y}), \chi) = 0 \quad \text{for all } y \in Y,$$

where $X_y := \pi^{-1}(y)$. Then

$$\text{Ext}_{G,k}^i(\Gamma_c(\mathcal{F}), \chi) = 0.$$

3.2 A projective generator

Write $H(G)$ for the Hecke algebra of G , namely $H(G) := \mathcal{S}(G) dg$, for a left invariant Haar measure dg on G . Denote by $\mathbb{C}_{G,X}$ the sheaf of $H(G)$ -valued locally constant functions on X . It is a G -equivariant ℓ -sheave under the diagonal action of G on $H(G) \times X$. Here G acts on $H(G)$ by the left translations, and the obvious identification

$$\bigsqcup_{x \in X} (\mathbb{C}_{G,X})_x = H(G) \times X$$

is used.

Denote by $Sh_G(X)$ the abelian category of G -equivariant ℓ -sheaves on X (a morphism in this category is a sheaf homomorphism $\mathcal{F} \rightarrow \mathcal{F}'$ so that the induced map $\bigsqcup_{x \in X} \mathcal{F}_x \rightarrow \bigsqcup_{x \in X} \mathcal{F}'_x$ is G -equivariant). Denote by $\mathcal{M}_X(G)$ the category of all smooth representations V of G equipped with a non-degenerate $\mathcal{S}(X)$ -module structure on it such that

$$g \cdot (\phi v) = (g \cdot \phi)(g \cdot v), \quad \text{for all } g \in G, \phi \in \mathcal{S}(X), v \in V.$$

Here the $\mathcal{S}(X)$ -module structure is non-degenerate means that

$$\mathcal{S}(X) \cdot V = V.$$

By [6, Proposition 1.14], Γ_c establishes an equivalence between the category of ℓ -sheaves on X and the category of non-degenerate $\mathcal{S}(X)$ -modules. This implies the following equivariant version of the localization theorem.

Proposition 3.1 *The functor*

$$\Gamma_c : Sh_G(X) \rightarrow \mathcal{M}_X(G) \tag{11}$$

is an equivalence of categories.

Lemma 3.4 *For each ℓ -space X , the sheaf \mathbb{C}_X is a projective object in the category of ℓ -sheaves on X .*

Proof By the equivalence of categories, we only need to show that $\mathcal{S}(X)$ is a projective object in the category of non-degenerate $\mathcal{S}(X)$ -modules. It is elementary and well known that X is a countable disjoint union of open compact subsets:

$$X = \bigsqcup_{i \in I} X_i.$$

Then $\mathcal{S}(X) = \bigoplus_{i \in I} \mathcal{S}(X_i)$ and the lemma easily follows. □

When X has only one element, the following proposition is proved by Blanc [5]. See also [7, Theorem A.4].

Proposition 3.5 *The G -equivariant ℓ -sheaf $\mathbb{C}_{G,X}$ is a projective generator in $Sh_G(X)$, that is, it is a projective object of $Sh_G(X)$, and for each G -equivariant ℓ -sheaf \mathcal{F} on X , there exist an epimorphism $\bigoplus_{i \in I} \mathbb{C}_{G,X} \rightarrow \mathcal{F}$ in $Sh_G(X)$ for some index set I .*

Proof By Proposition 3.1, we only need to show that $\mathcal{S}(X) \otimes H(G)$ is a projective generator in $\mathcal{M}_X(G)$. Here G acts on $\mathcal{S}(X) \otimes H(G)$ diagonally, and $\mathcal{S}(X)$ acts on $\mathcal{S}(X) \otimes H(G)$ through the multiplication on $\mathcal{S}(X)$.

For every $V \in \mathcal{M}_X(G)$, the linear map

$$\mathcal{S}(X) \otimes H(G) \otimes V \rightarrow V, \quad \phi \otimes \eta \otimes v \mapsto \phi \cdot (\eta.v),$$

is an epimorphism in $\mathcal{M}_X(G)$, where G and $\mathcal{S}(X)$ act on $\mathcal{S}(X) \otimes H(G) \otimes V$ through their action on $\mathcal{S}(X) \otimes H(G)$. This proves the second assertion of the proposition.

Now we show that $\mathcal{S}(X) \otimes H(G)$ is projective. Fix an element $\eta_0 \in \mathcal{S}(G)$ so that

$$\int_G \eta_0(g) d_r g = 1,$$

where $d_r g$ denotes a fixed right invariant Haar measure G .

Let

$$\begin{array}{ccc} & \mathcal{S}(X) \otimes H(G) & \\ & \downarrow F & \\ U & \xrightarrow{P} & V \longrightarrow 0 \end{array}$$

be a diagram in $\mathcal{M}_X(G)$ so that the map P is surjective. Lemma 3.4 implies that there exists a $\mathcal{S}(X)$ -module homomorphism $F' : \mathcal{S}(X) \otimes H(G) \rightarrow U$ which is a lifting of F , that is, $P \circ F' = F$. Define a linear map

$$F'' : \mathcal{S}(X) \otimes H(G) \rightarrow U, \quad \phi \otimes \omega \mapsto \int_G g^{-1} \cdot F'(g \cdot \phi \otimes (\eta_0 \cdot (g \cdot \omega))) d_r g.$$

Then it is routine to check that F'' is a well-define morphism in $\mathcal{M}_X(G)$ which lifts F . This finishes the proof. □

Corollary 3.6 *The functor $\Gamma_c : Sh_G(X) \rightarrow \mathcal{M}(G)$ is exact and maps projective objects to projective objects.*

Proof The functor is exact since (11) is an equivalence of categories. Proposition 3.5 implies that every projective object in $Sh_G(X)$ is isomorphic to a direct summand of $\bigoplus_{i \in I} \mathbb{C}_{G,X}$ for some index set I . Lemma 2.8 implies that as representations of G , $\Gamma_c(\mathbb{C}_{G,X}) = \mathcal{S}(X) \otimes H(G)$ is isomorphic to a direct sum of copies of $H(G)$. As a special case of Proposition 3.5, we know that $H(G)$ is a projective object in $\mathcal{M}(G)$. This proves that the functor Γ_c maps projective objects to projective objects. □

Corollary 3.7 *Let $Z \subset X$ be a G -stable locally closed subset of X . Then the functor*

$$Sh_G(X) \rightarrow Sh_G(Z), \quad \mathcal{F} \mapsto \mathcal{F}|_Z$$

is exact and maps projective objects to projective objects.

Proof Since $\mathbb{C}_{G,X|Z} = \mathbb{C}_{G,Z}$, the corollary follows by the argument as in the proof of Corollary 3.6. □

3.3 The proof of Theorem 3.2

Let \mathcal{F} be a G -equivariant ℓ -sheaf on X as in Theorem 3.2. For each smooth representation V of G , $\mathcal{F} \otimes V$ is clearly a G -equivariant ℓ -sheaf on X . Moreover, we have that

$$\Gamma_c(\mathcal{F} \otimes V) = \Gamma_c(\mathcal{F}) \otimes V \tag{12}$$

as a smooth representation of G .

Let Y and $\pi : X \rightarrow Y$ be as in Theorem 3.2. Note that $\Gamma_c(\mathcal{F})$ is a $C^\infty(X)$ -module, where $C^\infty(X)$ denotes the algebra of all \mathbb{C} -valued locally constant functions on X . The pull-back through π yields an algebra homomorphism $\mathcal{S}(Y) \rightarrow C^\infty(X)$. Using this homomorphism, we view $\Gamma_c(\mathcal{F})$ as a non-degenerate $\mathcal{S}(Y)$ -module. For each smooth representation V of G , recall that its co-invariant space is defined to be

$$V_G := \frac{V}{\text{span}\{g.v - v \mid g \in G, v \in V\}}.$$

For each non-degenerate $\mathcal{S}(Y)$ -module M and each $y \in Y$, denote by M_y the stalk at y of the ℓ -sheaf \tilde{M} on Y associated to M . To be explicit,

$$M_y := M \otimes_{\mathcal{S}(Y)} \mathbb{C}_y,$$

where \mathbb{C}_y denotes the ring \mathbb{C} with the evaluation map $\mathcal{S}(Y) \rightarrow \mathbb{C}$ at y .

The following proposition is proved in [6, Proposition 2.36].

Proposition 3.8 *The coinvariant space $(\Gamma_c(\mathcal{F}))_G$ is a non-degenerate $\mathcal{S}(Y)$ -module. Moreover, for each $y \in Y$, one has a natural vector space isomorphism*

$$((\Gamma_c(\mathcal{F}))_G)_y \cong (\Gamma_c(\mathcal{F}|_{X_y}))_G \quad (X_y := \pi^{-1}(y)).$$

Now we come to the proof of Theorem 3.2. In view of Lemma 2.15 and the equality (12), replacing \mathcal{F} by $\mathcal{F} \otimes V_1 \otimes V_2$, we only need to show that

$$H_i(G, \Gamma_c(\mathcal{F})) = 0, \tag{13}$$

under the assumption that

$$H_i(G, \Gamma_c(\mathcal{F}|_{X_y})) = 0 \quad \text{for all } y \in Y. \tag{14}$$

Take a projective resolution $\mathcal{P}_\bullet \rightarrow \mathcal{F}$ of \mathcal{F} in the category $Sh_G(X)$. By Corollaries 3.6 and 3.7, for all $y \in Y$, $\Gamma_c(\mathcal{P}_\bullet|_{X_y}) \rightarrow \Gamma_c(\mathcal{F}|_{X_y})$ is a projective resolution of $\Gamma_c(\mathcal{F}|_{X_y})$ in the category $\mathcal{M}(G)$. By the assumption of (14), the complex $(\Gamma_c(\mathcal{P}_\bullet|_{X_y}))_G$ is exact at degree i . Applying Proposition 3.8, we know that the complex $((\Gamma_c(\mathcal{P}_\bullet))_G)_y$ is exact at degree i . Therefore $(\Gamma_c(\mathcal{P}_\bullet))_G$ is exact at degree i as a complex of non-degenerate $\mathcal{S}(Y)$ -modules. (Recall that the category of ℓ -sheaves on Y is equivalent to the category of non-degenerate $\mathcal{S}(Y)$ -modules.) This proves (13) since by Corollary 3.6, $\Gamma_c(\mathcal{P}_\bullet) \rightarrow \Gamma_c(\mathcal{F})$ is a projective resolution of $\Gamma_c(\mathcal{F})$ in the category $\mathcal{M}(G)$.

4 A theorem of automatic extensions

4.1 Frobenius reciprocity and Shapiro’s lemma

Let H be a closed subgroup of an ℓ -group G . Then there is a unique character $\delta_{H \setminus G}$ of H such that

$$\text{Hom}_G \left(\text{ind}_H^G \delta_{H \setminus G}^{-1}, \mathbb{C} \right) \neq 0. \tag{15}$$

Here ind_H^G indicate the un-normalized Schwartz induction as in Sect. 2.4. The space (15) is then one-dimensional.

Let V be a smooth representation of G and let V_0 be a smooth representation of H . Recall the following well-known Frobenius reciprocity.

Lemma 4.1 *Fix a generator of the space (15). Then there is a canonical linear isomorphism*

$$\text{Hom}_G \left(\text{ind}_H^G V_0, V^\vee \right) \cong \text{Hom}_H \left(\delta_{H \setminus G} \otimes V_0, (V|_H)^\vee \right).$$

Combining Lemmas 2.7, 2.8 and 4.1, we get the following proposition.

Proposition 4.2 *Fix a generator of the space (15). Then there is a canonical linear isomorphism*

$$\text{Hom}_{G,k} \left(\text{ind}_H^G V_0, V^\vee \right) \cong \text{Hom}_H \left(\delta_{H \setminus G} \otimes J_{G,k} \otimes V_0, (V|_H)^\vee \right).$$

It is well known that Schwartz inductions preserve projectiveness, as in the following lemma.

Lemma 4.3 *If V_0 is projective as a smooth representations of H , then the smooth representation $\text{ind}_H^G V_0$ of G is also projective.*

Proof As in the proof of Lemma 2.9, V_0 is isomorphic to a direct summand of $\text{ind}_{\{1\}}^H V_0$. Therefore $\text{ind}_H^G V_0$ is isomorphic to a direct summand of

$$\text{ind}_H^G \left(\text{ind}_{\{1\}}^H V_0 \right) \cong \text{ind}_{\{1\}}^G V_0.$$

By [7, Theorem A.4], $\text{ind}_{\{1\}}^G V_0$ is projective. Therefore $\text{ind}_H^G V_0$ is also projective. \square

We have the following Shapiro’s lemma for generalized extensions.

Proposition 4.4 *Fix a generator of the space (15). Then there is a canonical linear isomorphism*

$$\text{Ext}_{G,k}^i \left(\text{ind}_H^G V_0, V^\vee \right) \cong \text{Ext}_H^i \left(\delta_{H \setminus G} \otimes J_{G,k} \otimes V_0, (V|_H)^\vee \right), \quad k \in \mathbb{N}, i \in \mathbb{Z}.$$

Proof Take a projective resolution $P_\bullet \rightarrow V_0$ of V_0 . Since “ind” is an exact functor, by Lemma 4.3, $\text{ind}_H^G P_\bullet \rightarrow \text{ind}_H^G V_0$ is also a projective resolution. Then $\text{Ext}_{G,k}^i(\text{ind}_H^G V_0, V^\vee)$ equals the i -th cohomology of the complex

$$\text{Hom}_{G,k} \left(\text{ind}_H^G P_\bullet, V^\vee \right).$$

The later is isomorphic to the complex

$$\text{Hom}_H \left(\delta_{H \setminus G} \otimes J_{G,k} \otimes P_\bullet, V_H^\vee \right)$$

by Proposition 4.2. By Lemma 2.9, $\delta_{H \setminus G} \otimes J_{G,k} \otimes P_\bullet \rightarrow \delta_{H \setminus G} \otimes J_{G,k} \otimes V_0$ is also a projective resolution in the category $\mathcal{M}(H)$. Therefore the proposition follows. \square

4.2 The case of homogeneous spaces

Let χ be a character of G . Note that there exists a natural isomorphism $\mathcal{S}(G/H) \simeq \text{ind}_H^G \mathbb{C}$ as representations of G via the following map

$$\phi \mapsto \left\{ g \mapsto \phi(g^{-1}) \right\},$$

for any $\phi \in \mathcal{S}(G/H)$. Hence as a special case of Proposition 4.4, we have the following proposition.

Proposition 4.5 *There is a linear isomorphism*

$$\text{Ext}_{G,k}^i \left(\mathcal{S}(G/H), \chi \right) \cong \text{Ext}_H^i \left(\delta_{H \setminus G} \otimes J_{G,k}, \chi \right), \quad k \in \mathbb{N}; i \in \mathbb{Z}.$$

Recall from the induction that G/H is said to be χ -admissible if $\text{Hom}_G(\mathcal{S}(G/H), \chi) \neq 0$. Proposition 4.5 implies that

$$G/H \text{ is } \chi\text{-admissible} \iff \chi|_H = \delta_{H \setminus G}. \tag{16}$$

Theorem 4.6 *Assume that H contains $H(F)$ as an open normal subgroup of finite index, where H is a connected linear algebraic group defined over F . If $G/H(F)$ is not χ -admissible, then*

$$\mathrm{Ext}_{G,k}^i(\mathcal{S}(G/H), \chi) = 0, \quad k = 0, 1, 2, \dots, \infty; \quad i \in \mathbb{Z}.$$

Proof When k is finite, this is a direct consequence of Proposition 4.5 and Corollary 2.13. Then for $k = \infty$, the theorem follows by Lemma 2.10. \square

4.3 The automatic extension theorem

Let G be a linear algebraic group over F , acting algebraically on an algebraic variety Z over F . We say that Z is homogeneous if the action of $G(\bar{F})$ on $Z(\bar{F})$ is transitive, where \bar{F} denotes an algebraic closure of F . The following result on homogeneous spaces over p -adic fields is well known.

Lemma 4.7 [21, Section 6.4, Corollary 2 and Section 3.1, Corollary 2] *If Z is homogeneous, then $Z(F)$ has only finitely many $G(F)$ -orbits, and every $G(F)$ -orbit is open in $Z(F)$.*

In general, recall the following result which is due to Rosenlicht [23]. See also [22, Theorem 4.4, p. 187].

Proposition 4.8 *There exists a G -stable open dense subvariety U of Z , a variety V over F , and a G -invariant morphism $f : U \rightarrow V$ of algebraic varieties over F such that for all F -rational point $y \in V$, the subvariety $f^{-1}(y)$ of U is homogeneous (under the action of G).*

Let χ be a character of $G(F)$ as in Theorem 1.4. Recall the notion of weakly χ -admissible from Definition 1.3.

Theorem 4.9 *Assume that every $G(F)$ -orbit in $Z(F)$ is not weakly χ -admissible. Then*

$$\mathrm{Ext}_{G(F),k}^i(\mathcal{S}(Z(F)), \chi) = 0, \quad \text{for all } k = 0, 1, 2, \dots, \infty; \quad i \in \mathbb{Z}.$$

Proof Using Lemma 2.10, we assume that k is finite. Using Proposition 4.8 inductively, and using the long exact sequences for extensions, we assume without loss of generality that Z equals the variety U of Proposition 4.8. Then the morphism f of Proposition 4.8 yields a $G(F)$ -invariant continuous map

$$f_0 : U(F) \rightarrow V(F).$$

By Corollary 3.3, we only need to show that

$$\mathrm{Ext}_{G(F),k}^i(\mathcal{S}(f_0^{-1}(y)), \chi) = 0.$$

for all $y \in V(F)$. This is obviously implied by Lemma 4.7 and Theorem 4.6. \square

We remark that Theorem 4.9 fails if the condition “not weakly χ -admissible” is replaced by the weaker condition “not χ -admissible”, even when \mathbf{G} is connected and reductive, and \mathbf{Z} is \mathbf{G} -homogeneous.

As in Theorem 1.4, let \mathbf{X} be an algebraic variety over \mathbf{F} on which \mathbf{G} acts algebraically, and let \mathbf{U} be a \mathbf{G} -stable open subvariety of \mathbf{X} . Using the long exact sequence for generalized extensions, Theorem 4.9 clearly implies the following automatic extension theorem, which contains Theorem 1.4 as a special case.

Theorem 4.10 *Assume that every $\mathbf{G}(\mathbf{F})$ -orbit in $(\mathbf{X} \setminus \mathbf{U})(\mathbf{F})$ is not weakly χ -admissible. Then for every $k = 0, 1, 2, \dots, \infty$ and every $i \in \mathbb{Z}$, the restriction map*

$$\text{Ext}_{\mathbf{G}(\mathbf{F}),k}^i(\mathcal{S}(\mathbf{X}(\mathbf{F})), \chi) \rightarrow \text{Ext}_{\mathbf{G}(\mathbf{F}),k}^i(\mathcal{S}(\mathbf{U}(\mathbf{F})), \chi)$$

is a linear isomorphism.

5 Semialgebraic spaces and meromorphic continuations

For the proof of Theorem 1.5, we describe a general form of the rationality of Igusa’s zeta integral, in the setting of semialgebraic geometry over p -adic fields. For the basics of p -adic semialgebraic geometry, we refer the readers to the following papers [8–10, 13, 14, 20].

5.1 Semialgebraic spaces

Recall that a subset of \mathbf{F}^n ($n \in \mathbb{N}$) is said to be semialgebraic if it is a finite Boolean combination of sets of the form

$$\left\{ x \in \mathbf{F}^n \mid f(x) = y^k \text{ for some } y \in \mathbf{F}^\times \right\},$$

where $f : \mathbf{F}^n \rightarrow \mathbf{F}$ is a polynomial function, and k is a positive integer. Given a semialgebraic subset X of \mathbf{F}^n and Y of \mathbf{F}^m ($m \in \mathbb{N}$), a map from X to Y is said to be semialgebraic if its graph is a semialgebraic subset of \mathbf{F}^{n+m} .

Let X be a set. We denote by A_X the set of all triples (U, U', ϕ) , U is a semialgebraic subset of \mathbf{F}^n for some $n \in \mathbb{N}$, U' is a subset of X , and $\phi : U \rightarrow U'$ is a bijection.

Definition 5.1 A semialgebraic structure over \mathbf{F} on a set X is a subset \mathcal{A} of A_X with the following properties:

- (a) every two elements (U_1, U'_1, ϕ_1) and (U_2, U'_2, ϕ_2) of \mathcal{A} are semialgebraically compatible, namely, the bijection

$$\phi_2^{-1} \circ \phi_1 : \phi_1^{-1}(U'_1 \cap U'_2) \rightarrow \phi_2^{-1}(U'_1 \cap U'_2)$$

has semialgebraic domain and codomain, and is semialgebraic;

- (b) there are finitely many elements (U_i, U'_i, ϕ_i) of \mathcal{A} , $i = 1, 2, \dots, r$ ($r \in \mathbb{N}$), such that

$$X = U'_1 \cup U'_2 \cup \dots \cup U'_r;$$

- (c) for every element of A_X , if it is semialgebraically compatible with all elements of \mathcal{A} , then itself is an element of \mathcal{A} .

A semialgebraic space over F (or a semialgebraic space for brevity) is defined to be a set together with a semialgebraic structure (over F) on it. By a semialgebraic chart of a semialgebraic space, we mean an element of the semialgebraic structure.

The following lemma is routine to check.

Lemma 5.2 *With the notation as in Definition 5.1, let*

$$\mathcal{A}_0 = \{(U_i, U'_i, \phi_i) \mid i = 1, 2, \dots, r\}$$

be a finite subset of A_X whose elements are pairwise semialgebraically compatible with each other. If

$$X = U'_1 \cup U'_2 \cup \dots \cup U'_r,$$

then the set of all elements in A_X which are semialgebraically compatible with all elements of \mathcal{A}_0 is a semialgebraic structure on X .

By Lemma 5.2, it is clear that the product of two semialgebraic spaces is naturally a semialgebraic space.

Definition 5.3 A subset S of a semialgebraic space X is said to be semialgebraic if $\phi^{-1}(S \cap U')$ is semialgebraic for every semialgebraic chart (U, U', ϕ) of X . A map from a semialgebraic space X to a semialgebraic space Y is said to be semialgebraic if its graph is semialgebraic in $X \times Y$.

It is clear that every semialgebraic subset of a semialgebraic space is naturally a semialgebraic space. Recall the following famous result of Macintyre [20].

Lemma 5.4 *Let $f : X \rightarrow Y$ be a semialgebraic map of semialgebraic spaces. Then for each semialgebraic subset S of X , $f(S)$ is a semialgebraic subset of Y .*

It is well-known and easy to see that Lemma 5.4 implies that the composition of two semialgebraic maps is semialgebraic, and the inverse image of a semialgebraic set under a semialgebraic map is semialgebraic. All semialgebraic spaces form a category whose morphisms are semialgebraic maps.

Definition 5.5 The dimension $\dim X$ of a semialgebraic space X is defined to be the largest non-negative integer n such that there is a semialgebraic subset of X which is isomorphic to a non-empty open semialgebraic subset of F^n as a semialgebraic space. By convention, the dimension of the empty set is defined to be $-\infty$.

The following proposition asserts that infinite semialgebraic spaces are classified by their dimensions.

Proposition 5.6 [9, Theorem 2] *Every infinite semialgebraic space X has positive dimension and is isomorphic to $\mathbb{F}^{\dim X}$.*

By a semialgebraic function on a semialgebraic space, we mean a semialgebraic map from it to \mathbb{F} .

Definition 5.7 A \mathbb{C} -valued function on a semialgebraic space X is said to be definable of order ≤ 0 if it belongs to the \mathbb{C} -algebra generated by the functions of the form

$$1_S, \quad |f|_{\mathbb{F}}^{s_0},$$

where $s_0 \in \mathbb{C}$, 1_S denotes the characteristic function of a semialgebraic subset S of X , and f is a nowhere vanishing semialgebraic function on X . It is said to be definable of order $\leq k$ ($k \geq 1$) if it is a linear combination of the functions of the form

$$\phi \cdot (\text{val} \circ g_1) \cdot (\text{val} \circ g_2) \cdots (\text{val} \circ g_k),$$

where ϕ is a definable function on X of order ≤ 0 , and g_1, g_2, \dots, g_k are nowhere vanishing semialgebraic functions on X .

Here $|\cdot|_{\mathbb{F}}$ denotes the normalized absolute value on \mathbb{F} , and $\text{val} : \mathbb{F}^\times \rightarrow \mathbb{Z}$ denotes the normalized valuation on \mathbb{F} .

5.2 Definable measures

Let us review some basic measure theory. Let X be a measurable space, that is, it is a set with a σ -algebra Σ on it, namely, Σ is a non-empty set of subsets of X which is closed under taking countable union and taking complement. An element of Σ is called a measurable subset of X . A non-negative measure on X is defined to be a map $\nu : \Sigma \rightarrow [0, \infty]$ which is countably additive, namely,

$$\nu \left(\bigsqcup_{i=1}^{\infty} S_i \right) = \sum_{i=1}^{\infty} \nu(S_i) \quad \text{for all pairwise disjoint elements } S_1, S_2, S_3, \dots \text{ of } \Sigma.$$

A complex function ϕ on X is said to be measurable if for each open subset U of \mathbb{C} , $\phi^{-1}(U) \in \Sigma$. Write $M(X)$ for the space of all measurable functions on X . We say that two elements of $M(X)$ are equal to each other almost everywhere with respect to ν if they are equal outside a set $S \in \Sigma$ with $\nu(S) = 0$.

Definition 5.8 A measure μ on X is a pair (ν, f) , where ν is a non-negative measure on X , and f is an element of

$$\frac{\{\phi \in M(X) \mid |\phi| \text{ equals } 1 \text{ almost everywhere with respect to } \nu\}}{\{\phi \in M(X) \mid \phi \text{ equals } 0 \text{ almost everywhere with respect to } \nu\}}.$$

Here the denominator is a vector space, and the numerator is a subset of $M(X)$ which is stable under translations by the denominator. Therefore the above quotient makes sense.

The non-negative measure ν of Definition 5.8 is called the total variation of $\mu = (\nu, f)$, and is denoted by $|\mu|$. For each measurable function ϕ on X , we say that the integral $\int_X \phi \mu$ converges if $\int_X |\phi| \nu < \infty$. In this case, the integration

$$\int_X \phi \mu := \int_X \phi f \nu$$

is a well-defined complex number. For each $Y \in \Sigma$, Y is a measurable space with the σ -algebra $\Sigma_Y := \{S \in \Sigma \mid S \subset Y\}$. We define the restriction $\mu|_Y$ of μ to Y in the obvious way. For each measurable function ϕ on X , the multiplication $\phi \mu$ is defined to be the measure $(|\phi| \nu, \frac{\phi}{|\phi|} f)$ on X .

Note that every semialgebraic space is naturally a measurable space: the σ -algebra is generated by all the semialgebraic subsets.

Definition 5.9 Let X be a semialgebraic space. A measure μ on X is said to be definable of order $\leq k$ ($k \in \mathbb{N}$) if there is a family $\{f_i : S_i \rightarrow X_i\}_{i=1,2,\dots,r}$ ($r \in \mathbb{N}$) of isomorphisms of semialgebraic spaces such that

- S_i is a semialgebraic subset of F^{n_i} , for some $n_i \in \mathbb{N}$ ($i = 1, 2, \dots, r$);
- $\{X_i\}_{i=1,2,\dots,r}$ is a cover of X by its semialgebraic subsets;
- for each $i = 1, 2, \dots, r$, the restriction of μ to S_i via f_i has the form $\phi_i \mu_{S_i}$, where μ_{S_i} denotes the restriction to S_i of a Haar measure of F^{n_i} , and ϕ_i is a definable function on S_i of order $\leq k$.

Write \mathbf{R} for the ring of integers in F . Fix a uniformizer $\varpi \in \mathbf{R}$. For each integer $k \geq 1$, put

$$\mathbf{R}_k := \bigsqcup_{r=0}^{\infty} \varpi^r (1 + \varpi^k \mathbf{R}).$$

Then \mathbf{R}_k is a semialgebraic set, since $\mathbf{R}_k = \{x \in F \mid x \neq 0, \text{ and } \text{ac}(x) \equiv 1 \pmod{\varpi^k}\}$ (see [13, Lemma 2.1(4)]), where $\text{ac}(x)$ is the annular component of x , i.e. $\text{ac}(x) = x \varpi^{-\text{ord}(x)}$. We say that a semialgebraic function f on $(\mathbf{R}_k)^n$ ($n \in \mathbb{N}$) is order monomial if there are integers d_1, d_2, \dots, d_n and an element $\beta \in F$ such that

$$|f(x_1, x_2, \dots, x_n)|_F = |\beta x_1^{d_1} x_2^{d_2} \dots x_n^{d_n}|_F \text{ for all } (x_1, x_2, \dots, x_n) \in (\mathbf{R}_k)^n.$$

The following theorem of rectilinearization with good Jacobians is proved in [10, Theorem 7].

Proposition 5.10 *Let X be a semi-algebraic set in F^n ($n \in \mathbb{N}$), and let $\{f_j\}_{j=1,\dots,r}$ ($r \in \mathbb{N}$) be a family of semialgebraic functions on X . Then there exists a family*

$$\{\phi_i : (\mathbf{R}_{k_i})^{n_i} \rightarrow F^n\}_{i=1,2,\dots,t} \quad (t \in \mathbb{N}, k_i \geq 1, n_i \in \mathbb{N})$$

of injective semialgebraic maps such that

- $\{\phi_i((\mathbf{R}_{k_i})^{n_i})\}_{i=1,2,\dots,t}$ forms a partition of X ;

- the restriction of f_j to $(\mathbb{R}_{k_i})^{n_i}$ through ϕ_i is order monomial ($j = 1, 2, \dots, r$, $i = 1, 2, \dots, t$); and
- for each $i = 1, 2, \dots, t$, if $n_i = n$, then ϕ_i is continuously differentiable and their Jacobian is order monomial.

We say that a measure μ on $(\mathbb{R}_k)^n$ is simple of order $\leq k$ if it is a linear combination of measures of the form

$$P(\text{val}(x_1), \text{val}(x_2), \dots, \text{val}(x_n))u_1^{\text{val}(x_1)}u_2^{\text{val}(x_2)} \dots u_n^{\text{val}(x_n)}\mu_{(\mathbb{R}_k)^n},$$

where P is a (complex) polynomial of degree $\leq k$, $u_1, u_2, \dots, u_n \in \mathbb{C}^\times$, and $\mu_{(\mathbb{R}_k)^n}$ is the restriction of a Haar measure on \mathbb{F}^n to $(\mathbb{R}_k)^n$. Clearly if a measure μ on $(\mathbb{R}_k)^n$ is simple of order $\leq k$, then μ is definable of order $\leq k$.

Lemma 5.11 *Every semialgebraic set of dimension $< n$ in \mathbb{F}^n has measure 0 with respect to a Haar measure on \mathbb{F}^n .*

Proof Let $S \subset \mathbb{F}^n$ be a non-empty semialgebraic set. Then there is a semialgebraic open subset S° of S such that S° is a locally closed and locally analytic submanifold of \mathbb{F}^n , and $\dim(S \setminus S^\circ) < \dim S$ (see [14] and [8, Section 1.2]). Note that the measure of S° is 0. Therefore the lemma follows by induction on $\dim S$. □

Proposition 5.10 and Lemma 5.11 easily imply the following proposition.

Proposition 5.12 *Let X be a semialgebraic space. Let $\{\mu_i\}_{i=1,2,\dots,r}$ ($r \in \mathbb{N}$) be a family of definable measures on X . Let $\{f_j\}_{j=1,2,\dots,s}$ ($s \in \mathbb{N}$) be a family of semialgebraic functions on X . Assume that μ_i has order $\leq d_i$ ($i = 1, 2, \dots, r$, $d_i \in \mathbb{N}$). Then there is a family $\{\phi_k : (\mathbb{R}_{m_k})^{n_k} \rightarrow X_k\}_{k=1,2,\dots,t}$ ($t \in \mathbb{N}$, $m_k \geq 1$, $n_k \in \mathbb{N}$) of isomorphisms of semialgebraic spaces such that*

- $\{X_k\}_{k=1,2,\dots,t}$ is a partition of X by its semialgebraic subsets;
- the restriction of μ_i to $(\mathbb{R}_{m_k})^{n_k}$ via ϕ_k is simple of order $\leq d_i$, and the restriction of f_j to $(\mathbb{R}_{m_k})^{n_k}$ via ϕ_k is order monomial, for all $k = 1, 2, \dots, t$; $i = 1, 2, \dots, r$; $j = 1, 2, \dots, s$.

5.3 Igusa zeta integrals

Write $q_{\mathbb{F}}$ for the cardinality of the residue field $\mathbb{R}/\varpi\mathbb{R}$. In this subsection, we prove the following general form of the convergence and rationality of Igusa zeta integrals.

Theorem 5.13 *Let μ be a definable measure of order $\leq k$ ($k \in \mathbb{N}$) on a semialgebraic space X . Let f be a nowhere vanishing bounded semialgebraic function on X such that*

$$|\mu|(X_{f,\epsilon}) < \infty \text{ for all } \epsilon > 0,$$

where $X_{f,\epsilon} := \{x \in X \mid |f(x)|_{\mathbb{F}} > \epsilon\}$. Then the integral

$$Z_{\mu}(f, s) := \int_X |f|_{\mathbb{F}}^s \mu \quad (s \in \mathbb{C})$$

converges when the real part of s is sufficiently large. Moreover, there exists a meromorphic function

$$M(s) = \frac{P(q_F^{-s}, q_F^s)}{(1 - a_1 q_F^{-s})^{n_1} (1 - a_2 q_F^{-s})^{n_2} \cdots (1 - a_r q_F^{-s})^{n_r}}$$

on \mathbb{C} , where

- $r \in \mathbb{N}$, a_1, a_2, \dots, a_r are pairwise distinct non-zero complex numbers;
- $n_1, n_2, \dots, n_r \in \{1, 2, \dots, \dim X + k\}$;
- P is a two variable polynomial with complex coefficients,

such that if $Z_\mu(f, s)$ is absolutely convergent for $s = s_0 \in \mathbb{C}$, then $M(s)$ is holomorphic at s_0 , and $Z_\mu(f, s_0) = M(s_0)$.

For each $k \in \mathbb{N}$, write $\mathcal{A}_k(\mathbb{Z}^n)$ ($n \in \mathbb{N}$) for the space of all complex functions which are linear combinations of the functions of the form

$$x \mapsto \chi(x)P(x),$$

where χ is a character of \mathbb{Z}^n , and P is a polynomial of degree $\leq k$.

By Proposition 5.12, in order to prove Theorem 5.13, we assume without loss of generality that $X = (\mathbb{R}_m)^n$ ($m \geq 1, n \in \mathbb{N}$), μ is simple of order $\leq k$, and f is order monomial. Then μ is the multiple of μ_X with a function

$$(x_1, x_2, \dots, x_n) \mapsto \phi(\text{val}(x_1), \text{val}(x_2), \dots, \text{val}(x_n)),$$

where μ_X denotes the restriction of the normalized Haar measure μ on \mathbb{R}^n (i.e. $\mu(\mathbb{R}^n) = 1$) to X , and $\phi \in \mathcal{A}_k(\mathbb{Z}^n)$. Since f is non-zero, bounded, and order monomial, we have that

$$|f(x_1, x_2, \dots, x_n)|_F = q_F^c |x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}|_F \quad \text{for all } (x_1, x_2, \dots, x_n) \in (\mathbb{R}_m)^n,$$

for some $c \in \mathbb{Z}$ and $d_1, d_2, \dots, d_n \in \mathbb{N}$. Then

$$\begin{aligned} Z_\mu(f, s) &= q_F^{cs - mn} \sum_{x=(x_1, x_2, \dots, x_n) \in \mathbb{N}^n} q_F^{-x_1} q_F^{-x_2} \cdots q_F^{-x_n} \phi(x) q_F^{-sd_1 x_1} q_F^{-sd_2 x_2} \cdots q_F^{-sd_n x_n}. \end{aligned}$$

Therefore Theorem 5.13 is implied by the following Proposition, which will be proved in the next subsection.

Proposition 5.14 *Let χ be a character on \mathbb{Z}^n of the form*

$$(x_1, x_2, \dots, x_n) \mapsto q_F^{-d_1 x_1} q_F^{-d_2 x_2} \cdots q_F^{-d_n x_n},$$

where $d_1, d_2, \dots, d_n \in \mathbb{N}$. Let $\phi \in \mathcal{A}_k(\mathbb{Z}^n)$ ($k \in \mathbb{N}$). Assume that

$$\sum_{x \in \mathbb{N}^n, |\chi(x)| > \epsilon} |\phi(x)| < \infty \text{ for all } \epsilon > 0. \tag{17}$$

Then the summation

$$Z_\phi(\chi, s) := \sum_{x \in \mathbb{N}^n} \phi(x)\chi(x)^s \quad (s \in \mathbb{C})$$

absolutely converges when the real part of s is sufficiently large. Moreover, there exists a meromorphic function

$$M(s) = \frac{P(q_{\mathbb{F}}^{-s})}{(1 - a_1 q_{\mathbb{F}}^{-s})^{n_1} (1 - a_2 q_{\mathbb{F}}^{-s})^{n_2} \cdots (1 - a_r q_{\mathbb{F}}^{-s})^{n_r}}$$

on \mathbb{C} , where

- $r \in \mathbb{N}$, a_1, a_2, \dots, a_r are pairwise distinct non-zero complex numbers;
- $n_1, n_2, \dots, n_r \in \{0, 1, 2, \dots, n + k\}$;
- P is a polynomial with complex coefficients,

such that if $Z_\phi(\chi, s)$ is absolutely convergent for $s = s_0 \in \mathbb{C}$, then $M(s)$ is holomorphic at s_0 , and $Z_\phi(\chi, s_0) = M(s_0)$.

5.4 Proof of Proposition 5.14

Write $\mathcal{A}(\mathbb{Z}^n) := \bigcup_{k=0}^\infty \mathcal{A}_k(\mathbb{Z}^n)$. We view the space $C(\mathbb{Z}^n)$ of \mathbb{C} -valued functions on \mathbb{Z}^n as a representation of \mathbb{Z}^n under translations:

$$(x_0.\phi)(x) := \phi(x + x_0), \quad x, x_0 \in \mathbb{Z}^n, \quad \phi \in C(\mathbb{Z}^n).$$

Lemma 5.15 *The space of \mathbb{Z}^n -finite vectors in $C(\mathbb{Z}^n)$ equals to $\mathcal{A}(\mathbb{Z}^n)$.*

Proof Let $C(\mathbb{Z}^n)^f$ be the space of all \mathbb{Z}^n -finite vectors. With respect to the action of \mathbb{Z}^n , we can decompose $C(\mathbb{Z}^n)^f$ into the direct sum of generalized eigenspaces,

$$C(\mathbb{Z}^n)^f = \bigoplus_{\chi} C(\mathbb{Z}^n)^\chi.$$

Here χ is taken over all characters of \mathbb{Z}^n , and $C(\mathbb{Z}^n)^\chi$ consists of functions ϕ such that for some $N > 0$,

$$(x_1 - \chi(x_1))(x_2 - \chi(x_2)) \cdots (x_N - \chi(x_N)).\phi = 0 \text{ for all } x_1, x_2, \dots, x_N \in \mathbb{Z}^n.$$

Then the space $\chi^{-1}C(\mathbb{Z}^n)^\chi$ exactly consists of the generalized invariant functions on \mathbb{Z}^n . It is well-known that the space of generalized invariant functions on \mathbb{Z}^n with

respect to the translations action coincide with the space of polynomials on \mathbb{Z}^n . This finishes the proof of the lemma. \square

Define $\mathcal{A}^\circ(\mathbb{Z}^n)$ to be the subspace of $\mathcal{A}(\mathbb{Z}^n)$ spanned by functions of the form

$$(x_1, x_2, \dots, x_n) \mapsto u_1^{x_1} u_2^{x_2} \cdots u_n^{x_n} P(x_1, x_2, \dots, x_n),$$

where u_1, u_2, \dots, u_n are non-zero complex numbers of absolute value < 1 , and P is a polynomial.

Lemma 5.16 *Let χ be a character on \mathbb{Z}^n of the form*

$$(x_1, x_2, \dots, x_n) \mapsto u_1^{x_1} u_2^{x_2} \cdots u_n^{x_n},$$

where u_1, u_2, \dots, u_t are complex numbers of absolute value 1, and $u_{t+1}, u_{t+2}, \dots, u_n$ are complex numbers of absolute value < 1 ($0 \leq t \leq n$). Let $\phi \in \mathcal{A}(\mathbb{Z}^n)$. Then

$$\sum_{x \in \mathbb{N}^n, |\chi(x)| > \epsilon} |\phi(x)| < \infty \text{ for all } \epsilon > 0 \tag{18}$$

if and only if $\phi \in \mathcal{A}^\circ(\mathbb{Z}^t) \otimes \mathcal{A}(\mathbb{Z}^{n-t})$.

Proof It is easy to see that (18) holds if and only if

$$\sum_{x \in \mathbb{N}^t} |\phi(x, x')| < \infty \text{ for all } x' \in \mathbb{N}^{n-t}. \tag{19}$$

Write A for the space of all $\phi \in \mathcal{A}(\mathbb{Z}^n)$ such that for some $x_0 \in \mathbb{Z}^t$ and $x'_0 \in \mathbb{Z}^{n-t}$,

$$\sum_{x \in x_0 + \mathbb{N}^t} |\phi(x, x')| < \infty \text{ for all } x' \in x'_0 + \mathbb{N}^{n-t}. \tag{20}$$

The space A is a \mathbb{Z}^n -subrepresentation of $\mathcal{A}(\mathbb{Z}^n)$ containing $\mathcal{A}^\circ(\mathbb{Z}^t) \otimes \mathcal{A}(\mathbb{Z}^{n-t})$.

Note that every one dimensional \mathbb{Z}^n -subrepresentation of A is contained in $\mathcal{A}^\circ(\mathbb{Z}^t) \otimes \mathcal{A}(\mathbb{Z}^{n-t})$. We also note that $\mathcal{A}^\circ(\mathbb{Z}^t) \otimes \mathcal{A}(\mathbb{Z}^{n-t})$ is closed under the multiplication by polynomials on \mathbb{Z}^n . It implies that $A \subset \mathcal{A}^\circ(\mathbb{Z}^t) \otimes \mathcal{A}(\mathbb{Z}^{n-t})$, by considering the generalized eigenspace decomposition of $\mathcal{A}(\mathbb{Z}^n)$ under the action of \mathbb{Z}^n . Hence the space A is exactly identical to $\mathcal{A}^\circ(\mathbb{Z}^t) \otimes \mathcal{A}(\mathbb{Z}^{n-t})$.

On the other hand, it is obvious that for all $\phi \in \mathcal{A}^\circ(\mathbb{Z}^t) \otimes \mathcal{A}(\mathbb{Z}^{n-t})$, (20) holds for all $x_0 \in \mathbb{Z}^t$ and $x'_0 \in \mathbb{Z}^{n-t}$, in particular (19) holds. This proves the lemma. \square

The following lemma is easy to check and we omit the details.

Lemma 5.17 *Let u be a non-zero complex number of absolute value < 1 . Then*

$$\sum_{x \in \mathbb{N}} \binom{x}{k} u^x = \frac{u^k}{(1-u)^{k+1}} \text{ for all } k \in \mathbb{N}.$$

Now we come to the proof of Proposition 5.14. Without loss of generality, assume that d_1, d_2, \dots, d_t are all 0, and $d_{t+1}, d_{t+2}, \dots, d_n$ are all positive ($0 \leq t \leq n$). By Lemma 5.16, the Assumption (17) implies that $\phi \in \mathcal{A}^\circ(\mathbb{Z}^t) \otimes \mathcal{A}(\mathbb{Z}^{n-t})$. Lemma 5.16 also implies that $Z_\phi(\chi, s)$ is absolutely convergent if and only if

$$\phi \chi^s \in \mathcal{A}^\circ(\mathbb{Z}^n) = \mathcal{A}^\circ(\mathbb{Z}^t) \otimes \mathcal{A}^\circ(\mathbb{Z}^{n-t}), \tag{21}$$

where χ^s denotes the character

$$\mathbb{Z}^n \rightarrow \mathbb{C}^\times, \quad (x_1, x_2, \dots, x_n) \mapsto q_F^{-sd_{t+1}x_{t+1}} q_F^{-sd_{t+2}x_{t+2}} \dots q_F^{-sd_n x_n}.$$

It is clear that (21) holds when the real part of s is large. This proves the first assertion of the proposition. Now assume that (21) holds. Without loss of generality, we further assume that

$$\phi(x_1, x_2, \dots, x_n) = \binom{x_1}{k_1} \binom{x_2}{k_2} \dots \binom{x_n}{k_n} q_F^{\alpha_1 x_1} q_F^{\alpha_2 x_2} \dots q_F^{\alpha_n x_n},$$

for all $x_1, x_2, \dots, x_n \in \mathbb{Z}^n$, where $\alpha_i \in \mathbb{C}, k_i \in \mathbb{N}$ ($i = 1, 2, \dots, n$) and $k_1 + k_2 + \dots + k_n \leq k$. Using Lemma 5.17, it is now easy to see that the summation $Z_\phi(\chi, s)$ has the desired property.

5.5 Semialgebraic ℓ -spaces and meromorphic continuations of distributions

Definition 5.18 A semialgebraic ℓ -space (over F) is a Hausdorff topological space X which is at the same time a semialgebraic space (over F), with the following property: there is a finite family of semialgebraic charts $\{(U_i, U'_i, \phi_i)\}_{i=1,2,\dots,r}$ ($r \in \mathbb{N}$) of X such that

- for all $i = 1, 2, \dots, r, U_i$ is locally closed in F^{n_i} for some $n_i \in \mathbb{N}, U'_i$ is open in X , and ϕ_i is a homeomorphism; and
- $X = U'_1 \cup U'_2 \cup \dots \cup U'_r$.

It is clear that every semialgebraic ℓ -space is an ℓ -space; the product of two semialgebraic ℓ -spaces is still a semialgebraic ℓ -space; and a locally closed semialgebraic subset of a semialgebraic ℓ -space is a semialgebraic ℓ -space. All semialgebraic ℓ -spaces form a category whose morphisms are semialgebraic continuous maps. For each algebraic variety X over $F, X(F)$ is obviously a semialgebraic ℓ -space. Note that every semialgebraic ℓ -space is naturally a measurable space, with the σ -algebra generated by all the open sets, which coincides with the one generated by all semialgebraic sets. An element of this σ -algebra is called a Borel subset of the semialgebraic ℓ -space.

Recall the following Riesz representation theorem.

Theorem 5.19 *Let X be a locally compact Hausdorff topological space which is locally secondly countable, namely, every point of X has a neighborhood which is secondly countable as a topological space. Then the map*

$$\{\text{locally finite measures on } X\} \rightarrow \{\text{continuous linear functionals on } C_c(X)\},$$

$$\mu \mapsto (\phi \mapsto \int_X \phi \mu)$$

is bijective.

Here $C_c(X)$ denotes the space of all compactly supported continuous functions on X , with the usual inductive topology. A measure μ on X is said to be locally finite if

$$|\mu|(K) < \infty \quad \text{for every compact subset } K \text{ of } X.$$

Recall that every locally finite measure on X is regular, as X is assumed to be locally secondly countable (see [11, Proposition 7.2.3]).

If X is an ℓ -space, then $\mathcal{S}(X)$ is a dense subspace of $C_c(X)$. Write $D(X) := \text{Hom}(\mathcal{S}(X), \mathbb{C})$ for the space of distributions on X . By Theorem 5.19, we have an embedding

$$\{\text{locally finite measure on } X\} \hookrightarrow D(X).$$

Using this embedding, we view every locally finite measure on X as a distribution on it.

Now let X be a semialgebraic ℓ -space and let f be a continuous semialgebraic function on X . Write $X_f := \{x \in X \mid f(x) \neq 0\}$, which is also a semialgebraic ℓ -space. Let μ be a locally finite definable measure on X_f of order $\leq k$ ($k \in \mathbb{N}$). Let $\phi \in \mathcal{S}(X)$. Note that ϕ is definable of order ≤ 0 . Theorem 5.13 implies that the integral

$$Z_{\mu,f}(\phi, s) := \int_{X_f} |f|_{\mathbb{F}}^s \phi \mu$$

defines a meromorphic function on \mathbb{C} . Moreover, for each $a_0 \in \mathbb{C}^\times$, $Z_{\mu,f}(\phi, s)$ is a rational function of $1 - a_0 q_{\mathbb{F}}^{-s}$. Therefore we have the Laurent expansion

$$Z_{\mu,f}(\phi, s) = \sum_{i \in \mathbb{Z}} Z_{\mu,f,a_0,i}(\phi) (1 - a_0 q_{\mathbb{F}}^{-s})^i.$$

We are interested in the distribution $Z_{\mu,f,a_0,i}$ on X . Note that $Z_{\mu,f,a_0,i} = 0$ when $i < -(\dim X + k)$.

5.6 The invariance property of $Z_{\mu,f,a_0,i}$

Let G be an abstract group which acts as automorphisms of the semialgebraic ℓ -space X . For every $g \in G$, and every distribution η on X (or on some G -stable locally closed subset of X), write $g * \eta$ for the push forward of η through the action of g . It is clear that for every distribution η on X , $\eta \in \text{Hom}_{G,k}(\mathcal{S}(X), \chi)$ if and only if

$$\left((g_0 - \chi(g_0^{-1}))(g_1 - \chi(g_1^{-1})) \cdots (g_k - \chi(g_k^{-1})) \right) * \eta = 0,$$

for all $g_0, g_1, \dots, g_k \in G$.

Now assume that there is a locally constant homomorphism $\chi_f : G \rightarrow \mathbb{Z}$ such that

$$|f(g \cdot x)|_F = q_F^{\chi_f(g)} |f(x)|_F, \quad g \in G, x \in X, \tag{22}$$

and there is a character χ_μ on G such that

$$\mu \in \text{Hom}_{G,k'}(\mathcal{S}(X_f), \chi_\mu), \quad \text{for some } k' \in \mathbb{N}.$$

Proposition 5.20 *Let $a_0 \in \mathbb{C}$. Let i_0 be an integer so that $Z_{\mu,f,a_0,i} = 0$ for all $i < i_0$. Then*

$$Z_{\mu,f,a_0,i} \in \text{Hom}_{G,k'+i-i_0}(\mathcal{S}(X), \chi_\mu a_0^{\chi_f}) \text{ for all } i \geq i_0.$$

Proof For each locally finite definable measure μ' on X_f , write $Z_{\mu'}(s)$ for the following distribution on X :

$$\phi \mapsto Z_{\mu',f}(\phi, s) := \int_{X_f} |f|_F^s \phi \mu'.$$

For each $i \in \mathbb{Z}$, write $Z_{\mu',i}$ for the i -th coefficients of the Laurent expansion of $Z_{\mu'}(s)$ as a rational function of $1 - a_0 q_F^{-s}$. It is a distribution on X .

The invariance property of $|f|_F$ implies that

$$(g - \chi_\mu(g^{-1})q_F^{-\chi_f(g)s}) * Z_\mu(s) = q_F^{-\chi_f(g)s} Z_{(g-\chi_\mu(g^{-1})) * \mu}, \quad \text{for all } g \in G.$$

Comparing the Laurent expansions of the two sides of the above equality, we know that

$$(g - \chi_\mu(g^{-1})a_0^{-\chi_f(g)s}) * Z_{\mu,i} - a_0^{-\chi_f(g)s} Z_{(g-\chi_\mu(g^{-1})) * \mu,i}$$

is a linear combination of distributions of the form $g' * Z_{\mu,i'}$, where $i' < i$ and $g' \in G$. Then the proposition follows by induction on $k' + i - i_0$. □

Corollary 5.21 *Let χ be a character of G . Then every generalized χ -invariant locally finite definable measure on X_f extends to a generalized χ -invariant distribution on X .*

Proof Write μ for the measure of the proposition. The distribution $Z_{\mu,f,1,0}$ on X extends μ and is generalized χ -invariant. □

6 Generalized invariant functions and definable measures

6.1 Generalized functions on homogeneous spaces

Let G be an ℓ -group and let X be a homogeneous space of it. We say that a distribution η on X is smooth if for every $x \in X$, there is an open compact subgroup K of G such

that $\eta|_{K \cdot x}$ is K -invariant. Denote by $D_c^\infty(X)$ the space of all smooth distributions on X with compact support. A generalized function on X is defined to be a linear functional on $D_c^\infty(X)$. The space of all generalized functions on X is denoted by $C^{-\infty}(X)$. As before, the space of all distributions on X is denoted by $D(X)$.

The following lemma is elementary and we omit its proof.

Lemma 6.1 *Let η be a smooth distribution on X which has non-zero restriction to all non-empty open subset of X . Then the map*

$$C^{-\infty}(X) \rightarrow D(X),$$

$$f \mapsto f\eta := (\phi \mapsto f(\phi\eta))$$

is a linear isomorphism.

Using the following injective linear map, we view every locally constant function on X as a generalized function on X :

$$C^\infty(X) \rightarrow C^{-\infty}(X),$$

$$f \mapsto (\eta \mapsto \eta(1_\eta f)),$$

where $C^\infty(X)$ is the space of locally constant functions on X , and 1_η denotes the characteristic function of the support of η .

Lemma 6.2 *Let K be an open compact subgroup of G . Then every K -invariant generalized function on X is a locally constant function on X .*

Proof Without loss of generality, assume that $G = K$. Then in view of Lemma 6.1, the lemma follows easily by the existence and uniqueness of K -invariant distributions on X . □

6.2 Characters on algebraic homogeneous spaces

Let \mathbf{G} be a linear algebraic group over F , with an algebraic subgroup \mathbf{H} of it. Denote by \mathbf{N} the unipotent radical of \mathbf{G} . Write $G := \mathbf{G}(F)$, $H := \mathbf{H}(F)$ and $N := \mathbf{N}(F)$.

In this subsection, we prove the following proposition.

Proposition 6.3 *Assume that \mathbf{G} is connected. Let χ be a character on G which is trivial on N and has finite order when restricted to H . Then χ has the form*

$$|\beta_1|_F^{s_1} \cdot |\beta_2|_F^{s_2} \cdot \dots \cdot |\beta_t|_F^{s_t} \cdot \chi_f,$$

where $t \in \mathbb{N}$, $\beta_1, \beta_2, \dots, \beta_t$ are algebraic characters on \mathbf{G} which are trivial on \mathbf{H} , $s_1, s_2, \dots, s_t \in \mathbb{C}$, and χ_f is a finite order character on G .

The following lemma is obvious.

Lemma 6.4 *Let \mathbf{A} be a split algebraic torus over F . Then every character on $\mathbf{A}(F)$ has the form*

$$|\beta_1|_F^{s_1} \cdot |\beta_2|_F^{s_2} \cdot \dots \cdot |\beta_t|_F^{s_t} \cdot \chi_f,$$

where $t \in \mathbb{N}$, $\beta_1, \beta_2, \dots, \beta_t$ are algebraic characters on \mathbf{A} , $s_1, s_2, \dots, s_t \in \mathbb{C}$, and χ_f is a finite order character on \mathbf{A} .

Generalizing Lemma 6.4, we have the following lemma.

Lemma 6.5 *Let \mathbf{A} be a split algebraic torus over F , with an algebraic subgroup \mathbf{S} of it. Let χ be a character on $\mathbf{A}(F)$ which has finite order when restricted to $\mathbf{S}(F)$. Then χ has the form*

$$|\beta_1|_F^{s_1} \cdot |\beta_2|_F^{s_2} \cdots |\beta_t|_F^{s_t} \cdot \chi_f,$$

where $t \in \mathbb{N}$, $\beta_1, \beta_2, \dots, \beta_t$ are algebraic characters on \mathbf{A}/\mathbf{S} , $s_1, s_2, \dots, s_t \in \mathbb{C}$, and χ_f is a finite order character on $\mathbf{A}(F)$.

Proof Denote by \mathbf{S}_0 the identity connected component of \mathbf{S} , which is also a split algebraic torus. Then there is an algebraic subtorus \mathbf{S}' of \mathbf{A} such that $\mathbf{A} = \mathbf{S}_0 \times_F \mathbf{S}'$. By Lemma 6.4, $\chi|_{\mathbf{S}'(F)}$ has the form

$$|\beta_1|_F^{s_1} \cdot |\beta_2|_F^{s_2} \cdots |\beta_t|_F^{s_t} \cdot \chi'_f,$$

where $t \in \mathbb{N}$, $\beta_1, \beta_2, \dots, \beta_t$ are algebraic characters on \mathbf{S}' , $s_1, s_2, \dots, s_t \in \mathbb{C}$, and χ'_f is a finite order character on $\mathbf{S}'(F)$. The group \mathbf{A}/\mathbf{S} is obviously identified with a quotient group of \mathbf{S}' , and there is a positive integer m such that $\beta_1^m, \beta_2^m, \dots, \beta_t^m$ descends to algebraic characters on \mathbf{A}/\mathbf{S} . Then we have that that

$$\chi = \chi|_{\mathbf{S}_0(F)} \otimes \left(|\beta_1^m|_F^{s_1/m} \cdot |\beta_2^m|_F^{s_2/m} \cdots |\beta_t^m|_F^{s_t/m} \cdot \chi'_f \right).$$

This proves the lemma. □

Lemma 6.6 *For each surjective algebraic homomorphism $\mathbf{G} \rightarrow \mathbf{G}'$ of linear algebraic groups over F , the image of the induced group homomorphism $\mathbf{G}(F) \rightarrow \mathbf{G}'(F)$ has finite index in $\mathbf{G}'(F)$.*

Proof This is a direct consequence of Lemma 4.7. □

Lemma 6.7 *Assume that \mathbf{G} is connected. Let \mathbf{A} be the largest central split torus in a Levi component \mathbf{L} of \mathbf{G} . Let χ be a character on G which is trivial on N . Then χ has finite order if and only if its restriction to $\mathbf{A}(F)$ has finite order.*

Proof The “only if” part of the lemma is trivial. We prove the “if” part. Assume that $\chi|_{\mathbf{A}(F)}$ has finite order. Let \mathbf{S} denote the simply connected covering of the derived subgroup of \mathbf{L} , let \mathbf{T} denote the maximal anisotropic central torus in \mathbf{L} . Then by Lemma 6.6, the image of the multiplication map

$$\varphi : (\mathbf{S}(F) \times \mathbf{T}(F) \times \mathbf{A}(F)) \rtimes N \rightarrow G = \mathbf{G}(F) \tag{23}$$

has finite index in G . Therefore it suffices to show that $\chi \circ \varphi$ has finite order. This holds because $\mathbf{S}(F)$ is a perfect group, $\mathbf{T}(F)$ is compact, $\chi|_{\mathbf{A}(F)}$ has finite order, and $\chi|_N$ is trivial. □

We are now ready to prove Proposition 6.3.

Proof Assume that \mathbf{G} is connected and let \mathbf{A} be as in Lemma 6.7. Let χ be as in Proposition 6.3. Write \mathbf{G}' for the largest quotient of \mathbf{G} which is a split algebraic torus. Consider the commutative diagram

$$\begin{array}{ccccc}
 \mathbf{A} & \xrightarrow{\varphi} & \mathbf{G} & \xrightarrow{\varphi'} & \mathbf{G}' \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathbf{S} := (\varphi' \circ \varphi)^{-1}(\mathbf{H}') & \longrightarrow & \varphi'^{-1}(\mathbf{H}') = \mathbf{H} \cdot \ker \varphi' & \longrightarrow & \mathbf{H}' := \varphi'(\mathbf{H}),
 \end{array}$$

where φ denotes the inclusion homomorphism, φ' denotes the quotient homomorphism, and the vertical arrows are inclusion homomorphisms. As in the proof of Lemma 6.7, we know that χ has finite order when restricted to $(\varphi'^{-1}(\mathbf{H}'))(\mathbb{F})$. In particular, $\chi|_{\mathbf{S}(\mathbb{F})}$ has finite order. By Lemma 6.5, there are algebraic characters $\beta_1, \beta_2, \dots, \beta_t$ ($t \in \mathbb{N}$) on \mathbf{A}/\mathbf{S} and $s_1, s_2, \dots, s_t \in \mathbb{C}$ so that the character

$$\chi|_{\mathbf{S}(\mathbb{F})} \cdot (|\beta_1|_{\mathbb{F}}^{s_1} \cdot |\beta_2|_{\mathbb{F}}^{s_2} \cdots |\beta_t|_{\mathbb{F}}^{s_t})^{-1},$$

on $\mathbf{S}(\mathbb{F})$ has finite order. Since $\mathbf{A}/\mathbf{S} = \mathbf{G}'/\mathbf{H}' = \mathbf{G}/(\mathbf{H} \cdot \ker \varphi')$, we may view $\beta_1, \beta_2, \dots, \beta_t$ as algebraic characters on \mathbf{G} which are trivial on \mathbf{H} . Therefore Proposition 6.3 follows by Lemma 6.7. □

Proposition 6.8 *Assume that \mathbf{G} is connected. Let $\chi' : \mathbf{G} \rightarrow \mathbb{C}$ be a locally constant group homomorphism which is trivial on \mathbf{H} . Then χ' is a linear combination of the characters of the form $\text{val} \circ \alpha$, where α is an algebraic characters on \mathbf{G} which are trivial on \mathbf{H} .*

Proof Note that \mathbb{C} has no nontrivial finite subgroup. This implies that $\chi'|_N$ is trivial. Then the proposition is proved by the same argument of the proof of Proposition 6.3 □

6.3 Generalized invariant functions on algebraic homogeneous spaces

We continue with the notation of the last subsection. Put $X := G/H$.

This subsection is to prove the following proposition.

Proposition 6.9 *Assume that \mathbf{G} is connected. Let χ be a character on G which is trivial on N . Then every non-zero element of $\text{Hom}_{G,k}(D_c^\infty(X), \chi)$ ($k \in \mathbb{N}$) is a smooth function on X of the form*

$$P(\text{val} \circ \alpha_1, \text{val} \circ \alpha_2, \dots, \text{val} \circ \alpha_r) \cdot |\beta_1|_{\mathbb{F}}^{s_1} \cdot |\beta_2|_{\mathbb{F}}^{s_2} \cdots |\beta_t|_{\mathbb{F}}^{s_t} \cdot \chi_f,$$

where $r, t \in \mathbb{N}$, $\alpha_1, \alpha_2, \dots, \alpha_r$ and $\beta_1, \beta_2, \dots, \beta_t$ are algebraic characters on \mathbf{G} which are trivial on \mathbf{H} , $s_1, s_2, \dots, s_t \in \mathbb{C}$, P is a polynomial of degree $\leq k$, and χ_f is a finite order character on G which is trivial on H such that

$$\chi = |\beta_1|_{\mathbb{F}}^{s_1} \cdot |\beta_2|_{\mathbb{F}}^{s_2} \cdots \cdots |\beta_r|_{\mathbb{F}}^{s_r} \cdot \chi_f.$$

First we have the following lemma.

Lemma 6.10 *Let χ be a character on G which is trivial on an open compact subgroup K of G . Then every element $f \in \text{Hom}_{G,k}(D_c^\infty(X), \chi)$ ($k \in \mathbb{N}$) is a K -invariant smooth function on X .*

Proof Lemma 2.3 implies that f is K -invariant. Therefore f is a smooth function by Lemma 6.2. □

Lemma 6.11 *Let χ be a character on G . If the space $\text{Hom}_{G,\infty}(D_c^\infty(X), \chi)$ is non-zero, then $\chi|_H$ is trivial.*

Proof Assume that $\text{Hom}_{G,\infty}(D_c^\infty(X), \chi) \neq 0$. Then $\text{Hom}_G(D_c^\infty(X), \chi) \neq 0$. By Lemma 6.10, we have a non-zero smooth function f on X such that

$$f(g \cdot x) = \chi(g)f(x), \quad \text{for all } g \in G, \quad x \in X.$$

This implies that $\chi|_H$ is trivial. □

For each free abelian group Λ of finite rank, we view the space $C(\Lambda)$ of all complex functions on Λ as a representation of Λ by left translations:

$$g \cdot f(x) := f(-g + x), \quad g, x \in \Lambda, \quad f \in C(\Lambda).$$

Lemma 6.12 *Let Λ be a free abelian group of finite rank, with a subgroup Λ_0 of it. Then every element in $C(\Lambda)^{\Lambda,k} \cap C(\Lambda)^{\Lambda_0}$ ($k \in \mathbb{N}$) has the form*

$$P(\lambda_1, \lambda_2, \dots, \lambda_r)$$

where $r \in \mathbb{N}$, $\lambda_1, \lambda_2, \dots, \lambda_r$ are group homomorphisms from Λ/Λ_0 to \mathbb{C} , and P is a polynomial of degree $\leq k$.

Proof Lemma 5.15 easily implies that

$$C(\Lambda)^{\Lambda,k} = \{\text{polynomial functions on } \Lambda \text{ of degree } \leq k\}, \quad (k \in \mathbb{N}).$$

Then it is elementary to see that the lemma holds. □

Lemma 6.13 *Assume that \mathbf{G} is connected. Then every element f of $\text{Hom}_{G,k}(D_c^\infty(X), \mathbb{C})$ ($k \in \mathbb{N}$) is a smooth function on X of the form*

$$P(\text{val} \circ \alpha_1, \text{val} \circ \alpha_2, \dots, \text{val} \circ \alpha_r)$$

where $r \in \mathbb{N}$, $\alpha_1, \alpha_2, \dots, \alpha_r$ are algebraic characters on \mathbf{G} which are trivial on \mathbf{H} , and P is a polynomial of degree $\leq k$.

Proof By Lemma 6.10, f is a G° -invariant function on X . We identify it with a function on Λ_G which is $[H]$ -invariant, where $[H]$ denotes the image of H under the quotient homomorphism $G \rightarrow \Lambda_G$. Then

$$f \in C(\Lambda_G)^{\Lambda_{G,k}} \cap C(\Lambda_G)^{[H]}.$$

Therefore the lemma follows by combining Lemma 6.12 and Proposition 6.8. □

Now we prove Proposition 6.9. Let f be a non-zero element of $\text{Hom}_{G,k}(D_c^\infty(X), \chi)$ ($k \in \mathbb{N}$). We view χ as a function on X since $\chi|_H$ is trivial by Lemma 6.11. Then

$$\chi^{-1} \cdot f \in \text{Hom}_{G,k}(D_c^\infty(X), \mathbb{C}).$$

Therefore Proposition 6.9 follows by combining Lemma 6.13 and Proposition 6.3.

6.4 Nash manifolds and volume forms

Definition 6.14 A Nash manifold (over F) is a locally analytic manifold X over F which is at the same time a semialgebraic space (over F) with the following property: there is a finite family of semialgebraic charts $\{(U_i, U'_i, \phi_i)\}_{i=1,2,\dots,r}$ ($r \in \mathbb{N}$) of X such that

- for all $i = 1, 2, \dots, r$, U_i is an open semialgebraic subset in F^{n_i} for some $n_i \geq 0$, U'_i is open in X , and ϕ_i is a locally analytic diffeomorphism; and
- $X = U'_1 \cup U'_2 \cup \dots \cup U'_r$.

All Nash manifolds form a category whose morphisms are Nash maps (namely, locally analytic semialgebraic maps). Every Nash manifold is clearly a semialgebraic ℓ -space. Let X be a Nash manifold. Then the tangent bundle

$$T(X) = \bigsqcup_{x \in X} T_x(X)$$

and the cotangent bundle

$$T^*(X) = \bigsqcup_{x \in X} T_x^*(X)$$

are both naturally Nash manifolds. Therefore

$$\wedge^{\text{top}} T^*(X) := \bigsqcup_{x \in X} \wedge^{\dim T_x(X)} T_x^*(X) \tag{24}$$

is also a Nash manifold. Consequently, for each $m \geq 1$, the line bundle $(\wedge^{\text{top}} T^*(X))^{\otimes m}$ is also a Nash manifold. By a Nash m -volume form on X , we mean a Nash section of the bundle $(\wedge^{\text{top}} T^*(X))^{\otimes m}$ over X . Fix a Haar measure μ_F on F . Attach to a Nash

m -volume form ω on X , we have a non-negative locally finite measure $|\omega|_{\mathbb{F}}^{\frac{1}{m}}$ on X as usual: in local coordinate, if

$$\omega = f(x_1, x_2, \dots, x_n)(dx_1 \wedge dx_2 \wedge \dots \wedge dx_n)^{\otimes m},$$

then

$$|\omega|_{\mathbb{F}}^{\frac{1}{m}} = |f(x_1, x_2, \dots, x_n)|_{\mathbb{F}}^{\frac{1}{m}} d\mu_{\mathbb{F}}(x_1) \otimes d\mu_{\mathbb{F}}(x_2) \otimes \dots \otimes d\mu_{\mathbb{F}}(x_n).$$

The following lemma is clear.

Lemma 6.15 *For each Nash m -volume form ω on X ($m \geq 1$), the measure $|\omega|_{\mathbb{F}}^{1/m}$ is locally finite and definable of order ≤ 0 .*

As usual, write \mathcal{O}_Y for the structure sheaf of an algebraic variety Y over \mathbb{F} . Let X be a smooth algebraic variety over \mathbb{F} . Let $\Omega_X := \Omega_{X/\mathbb{F}}$ denote the sheaf of algebraic differential forms on X . Similar to (24), we define $\wedge^{\text{top}}\Omega_X$, which is a locally free \mathcal{O}_X -module of rank 1. By an algebraic m -volume form on X , we mean a global section of the sheaf $(\wedge^{\text{top}}\Omega_X)^{\otimes m}$ over X . The notion of algebraic 1-volume form exactly coincides with the usual notion of algebraic volume form.

Given an algebraic m -volume form ω on X , a Nash m -volume form on the Nash manifold $X(\mathbb{F})$ is obviously associated to it. We define $|\omega|_{\mathbb{F}}^{\frac{1}{m}}$ to be the non-negative locally finite measure on $X(\mathbb{F})$ attach to this Nash m -volume form.

6.5 Generalized invariant distributions on algebraic homogeneous spaces

We continue with the notation of Sects. 6.2 and 6.3. Then $X = G/H$ is naturally a Nash manifold since it is a semialgebraic open subset of $(G/H)(\mathbb{F})$. In this subsection, we prove the following theorem.

Theorem 6.16 *Assume that G is connected. Let χ be a character on G which is trivial on N . Assume that χ is admissible on X . Then every element of $\text{Hom}_{G,k}(\mathcal{S}(X), \chi)$ ($k \in \mathbb{N}$) is a measure on X and is of the form*

$$P(\text{val} \circ \alpha_1, \text{val} \circ \alpha_2, \dots, \text{val} \circ \alpha_r) \cdot |\beta_1|_{\mathbb{F}}^{s_1} \cdot |\beta_2|_{\mathbb{F}}^{s_2} \cdot \dots \cdot |\beta_t|_{\mathbb{F}}^{s_t} \cdot \chi_f \cdot (|\omega|_{\mathbb{F}}^{1/m})|_X,$$

where $r, t \in \mathbb{N}$, $\alpha_1, \alpha_2, \dots, \alpha_r$ and $\beta_1, \beta_2, \dots, \beta_t$ are algebraic characters on G which are trivial on H , $s_1, s_2, \dots, s_t \in \mathbb{C}$, P is a polynomial of degree $\leq k$, χ_f is a finite order character on G which is trivial on H , $m \geq 1$, and ω is an algebraic m -volume form on G/H which is δ -invariant for some algebraic character δ of G defined over \mathbb{F} with the property that

$$\chi = |\beta_1|_{\mathbb{F}}^{s_1} \cdot |\beta_2|_{\mathbb{F}}^{s_2} \cdot \dots \cdot |\beta_t|_{\mathbb{F}}^{s_t} \cdot \chi_f \cdot |\delta|_{\mathbb{F}}^{\frac{1}{m}}.$$

Here the algebraic m -volume form ω on G/H is δ -invariant means that

$$g \cdot \omega_{\mathbb{F}} = \delta(g^{-1})\omega_{\mathbb{F}}, \quad \text{for all } g \in G(\bar{\mathbb{F}}),$$

where $\omega_{\bar{F}}$ denotes the base extension to \bar{F} of ω . We also say that an algebraic m -volume form ω is semi-invariant if it is δ -invariant for some algebraic character δ of G defined over F .

For the proof of Theorem 6.16, in the rest of this subsection, we assume that G is connected. We start with the following lemma.

Lemma 6.17 *Let δ be an algebraic character on H (defined over F). If the character $|\delta|_F : H \rightarrow \mathbb{C}^\times$ extends to a character on G , then δ^m extends to an algebraic character on G for some positive integer m .*

Proof We first assume that H is also connected. Write Ψ_G and Ψ_H for the groups of algebraic characters of G and H , respectively. They are free abelian groups of finite rank. We identify the group of positive characters on G with $\Psi_G \otimes_{\mathbb{Z}} \mathbb{R}$ via the following isomorphism:

$$\Psi_G \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \{\text{positive characters on } G\}, \quad \delta \otimes a \mapsto |\delta|_F^a.$$

Likewise we identify the group of positive characters on H with $\Psi_H \otimes_{\mathbb{Z}} \mathbb{R}$. Then we have a commutative diagram

$$\begin{array}{ccc} \Psi_G & \longrightarrow & \Psi_G \otimes_{\mathbb{Z}} \mathbb{R} \\ \alpha_1 \downarrow & & \downarrow \alpha_2 \\ \Psi_H & \longrightarrow & \Psi_H \otimes_{\mathbb{Z}} \mathbb{R}, \end{array}$$

where α_1 denotes the map of restrictions of algebraic characters, and α_2 denotes the map of restrictions of positive characters.

Now let $\delta \in \Psi_H$ and assume that $|\delta|_F$ extends to a character on G . Then $|\delta|_F$ extends to a positive character on G , that is, it belongs to the image of α_2 . Then it is elementary that

$$|\delta|_F \in \alpha_2(\Psi_G \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Therefore $|\delta^m|_F \in \alpha_2(\Psi_G)$ for some positive integer m . Then $\delta^m \in \alpha_1(\Psi_G)$ and the lemma is proved in the case when H is connected.

Now we drop the assumption that H is connected. Let $\delta \in \Psi_H$ and assume that $|\delta|_F$ extends to a character on G as before. We have proved that there exists an algebraic character δ' on G such that

$$\delta'|_{H_0} = (\delta|_{H_0})^m$$

for some positive integer m , where H_0 denotes the identity connected component of H . Then

$$\delta'^d|_H = \delta^{md},$$

where d denotes the cardinality of the group $(H/H_0)(\bar{F})$, and \bar{F} denotes an algebraic closure of F . This finishes the proof of the Lemma. □

Lemma 6.18 *Assume that X is χ -admissible for some character χ on G . Then there is an algebraic character δ of \mathbf{G} and a positive integer m such that there is a non-zero algebraic m -volume form ω on \mathbf{G}/\mathbf{H} which is δ -invariant.*

Proof Let $\Delta_{\mathbf{G}}$ denote the algebraic modular character of \mathbf{G} , namely the determinant of the adjoint representation of \mathbf{G} on the Lie algebra $\text{Lie}(\mathbf{G})$. Likewise let $\Delta_{\mathbf{H}}$ denote the algebraic modular character of \mathbf{H} . Put $\Delta_{\mathbf{G}/\mathbf{H}} := \frac{\Delta_{\mathbf{G}}}{\Delta_{\mathbf{H}}}$, which is an algebraic character on \mathbf{H} . Recall the character $\delta_{H \setminus G}$ on H from Sect. 4.1. Note that

$$\delta_{H \setminus G} = |\Delta_{\mathbf{G}/\mathbf{H}}|_{\mathbf{F}}. \tag{25}$$

Assume that X is χ -admissible for some character χ on G . By (16), the character $\delta_{H \setminus G}$ extends to a character on G . Then lemma 6.17 implies that the algebraic character $\Delta_{\mathbf{G}/\mathbf{H}}^m$ extends to an algebraic character δ on \mathbf{G} , for some positive integer m . Then the lemma follows by the algebraic version of Frobenius reciprocity. \square

Now we come to the proof of Theorem 6.16. Let $\eta \in \text{Hom}_{G,k}(\mathcal{S}(X), \chi)$. We assume that η is non-zero. Then X is χ -admissible. By Lemma 6.18, there is an algebraic character δ on \mathbf{G} , a positive integer m , and a non-zero algebraic m -volume form ω on \mathbf{G}/\mathbf{H} which is δ -invariant. By Lemma 6.1, there is a unique generalized function $f \in C^{-\infty}(X)$ such that $\eta = f|\omega|_{\mathbf{F}}^{1/m}$. Then

$$f \in \text{Hom}_{G,k} \left(D_c^{\infty}(X), \chi|\delta|_{\mathbf{F}}^{\frac{-1}{m}} \right).$$

Therefore Theorem 6.16 follows by Proposition 6.9.

6.6 Definability of generalized invariant distributions

First we have the following elementary lemma.

Lemma 6.19 *Every finite index subgroup of \mathbf{F}^{\times} is semialgebraic. Consequently, every finite order character on \mathbf{F}^{\times} is definable of order ≤ 0 .*

Proof Every subgroup of \mathbf{F}^{\times} of finite index $m \geq 1$ contains $(\mathbf{F}^{\times})^m$. Since $(\mathbf{F}^{\times})^m$ is a semialgebraic subgroup of \mathbf{F}^{\times} of finite index, the lemma follows. \square

Recall the following semi-algebraic selection theorem of [24]. See also [14, Appendix].

Lemma 6.20 *Every surjective semialgebraic map of semialgebraic spaces has a semi-algebraic section.*

We continue with the notation of the last subsection, but drop the assumption that \mathbf{G} is connected. Generalizing Lemma 6.19, we have the following lemma.

Lemma 6.21 *Every finite order character $\chi_{\mathbf{f}}$ on G is definable of order ≤ 0 .*

Proof Recall the multiplication map

$$\varphi : (\mathbf{S}(\mathbb{F}) \times \mathbf{T}(\mathbb{F}) \times \mathbf{A}(\mathbb{F})) \ltimes N \rightarrow G = \mathbf{G}(\mathbb{F})$$

from (23). Since the image of φ is a semialgebraic subgroup of G of finite index. By Lemma 6.20, it suffices to show that the finite order character $\chi'_f := \chi_f \circ \varphi$ is definable of order ≤ 0 . This is true because χ'_f has trivial restriction to $\mathbf{S}(\mathbb{F})$ and N , $(\chi'_f)|_{\mathbf{T}(\mathbb{F})}$ is a Bruhat-Schwartz function, and by Lemma 6.19, $(\chi'_f)|_{\mathbf{A}(\mathbb{F})}$ is definable of order ≤ 0 . \square

Proposition 6.22 *Let χ be a character on G which is trivial on N . Then every element of $\text{Hom}_{G,k}(\mathcal{S}(X), \chi)$ ($k \in \mathbb{N}$) is a definable measure on X of order $\leq k$.*

Proof Without loss of generality, assume that \mathbf{G} is connected. Then the proposition follows by Theorem 6.16 and Lemma 6.21. \square

6.7 Locally finiteness of some algebraic measures

For each algebraic variety \mathbf{X} over \mathbb{F} , write \mathbf{X}_{sm} for the smooth part of \mathbf{X} , which is an open subvariety of \mathbf{X} . Recall that a strong resolution of singularities of \mathbf{X} is a smooth algebraic variety $\tilde{\mathbf{X}}$ (over \mathbb{F}) together with a proper birational morphism $\pi : \tilde{\mathbf{X}} \rightarrow \mathbf{X}$ such that $\pi : \pi^{-1}(\mathbf{X}_{\text{sm}}) \rightarrow \mathbf{X}_{\text{sm}}$ is an isomorphism. The famous theorem of Hironaka says that \mathbf{X} always has a strong resolution of singularities.

Recall the following definition.

Definition 6.23 We say that an algebraic variety \mathbf{X} over \mathbb{F} has rational singularities if it is normal, and there exists a strong resolution of singularities $\pi : \tilde{\mathbf{X}} \rightarrow \mathbf{X}$ of \mathbf{X} such that the higher derived direct images vanish, that is, $\mathcal{R}^i \pi_* (\mathcal{O}_{\tilde{\mathbf{X}}}) = 0$ for all $i > 0$. Here $\mathcal{R}^i \pi_*$ denotes the i -th derived functor of the push-forward functor of sheaves via π .

We will use the following property of algebraic varieties with rational singularities.

Lemma 6.24 *Let \mathbf{X} be an algebraic variety over \mathbb{F} with rational singularities. Let \mathbf{U} be a smooth open subvariety of \mathbf{X} whose complement has codimension ≥ 2 . Let $\pi : \tilde{\mathbf{X}} \rightarrow \mathbf{X}$ be a strong resolution of singularities. Then for each algebraic volume form ω on \mathbf{U} , there is an algebraic volume form $\tilde{\omega}$ on $\tilde{\mathbf{X}}$ so that its restriction to $\pi^{-1}(\mathbf{U})$ is identical to ω via the isomorphism $\pi : \pi^{-1}(\mathbf{U}) \rightarrow \mathbf{U}$.*

Proof See [19, P.50, Proposition] or [2, Proposition 1.4]. \square

Given a measurable space X with a measurable subset Y of it, for each measure μ on Y , we write $\mu|_X$ for the measure on X which is obtained from μ by the extension by zero.

Proposition 6.25 (cf. [1, Lemma 3.4.1]) *Let \mathbf{X} be an algebraic variety over \mathbb{F} with rational singularities. Let \mathbf{U} be a smooth open subvariety of \mathbf{X} whose complement has codimension ≥ 2 . Then the measure $(|\omega|_{\mathbb{F}})|^{\mathbf{X}(\mathbb{F})}$ is locally finite for all algebraic volume form ω on \mathbf{U} .*

Proof Let $\pi : \tilde{X} \rightarrow X$ be a strong resolution of singularities. Let $\tilde{\omega}$ be as in Lemma 6.24. Write $\pi(|\tilde{\omega}|)$ for the push-forward of $|\tilde{\omega}|_F$ through the map

$$\pi : \tilde{X}(F) \rightarrow X(F). \tag{26}$$

Then the measure $\pi(|\tilde{\omega}|)$ is locally finite since (26) is a proper continuous map of topological spaces. The proposition then follows by noting that $\pi(|\tilde{\omega}|) - (|\omega|_F)^{X(F)}$ is a non-negative measure. □

Definition 6.26 We say that an algebraic variety X over F has Gorenstein rational singularities if it has rational singularities, and the push forward \mathcal{K}_X of $\wedge^{\text{top}}\Omega_{X_{\text{sm}}}$ through the inclusion map $X_{\text{sm}} \hookrightarrow X$ is a locally free \mathcal{O}_X -module.

The sheaf \mathcal{K}_X is called the dualizing sheaf of X . If X is smooth, then $\mathcal{K}_X \simeq \wedge^{\text{top}}\Omega_X$. We have the following examples of Gorenstein rational singularities:

- (1) If X has symplectic singularities, then X has rational Gorenstein singularities, see [4, Proposition 1.3].
- (2) The normalization of nilpotent varieties in semisimple Lie algebras have Gorenstein rational singularities, see [17].

Recall the following standard fact in algebraic geometry.

Lemma 6.27 *Let X be a normal variety and suppose that \mathcal{F} is a locally free sheaf on X . Let U be an open subvariety of X such that the complement $X \setminus U$ is of codimension ≥ 2 . Then the restriction map $\Gamma(X, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F})$ is an isomorphism.*

Proof See [16, Proposition 1.11, Theorem 1.9]. □

We say that a quasi-coherent sheaf \mathcal{F} on an algebraic variety X is torsion-free if for every $x \in X$, the stalk \mathcal{F}_x is torsion-free as a module of the local ring $\mathcal{O}_{X,x}$. The following fact is standard. We omit its easy proof.

Lemma 6.28 *Let \mathcal{F} be a torsion-free quasi-coherent sheaf on an algebraic variety X . Let U be an open subset of X whose complement has codimension ≥ 1 . Then the restriction map $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ is injective.*

Similar to Lemma 6.24, we have the following proposition.

Proposition 6.29 *Let X be an algebraic variety over F with Gorenstein rational singularities. Let U be a smooth open subvariety of X whose complement has codimension ≥ 2 . Let $\pi : \tilde{X} \rightarrow X$ be a strong resolution of singularities. Then for every positive integer m and every algebraic m -volume form ω on U , there is an algebraic m -volume form $\tilde{\omega}$ on \tilde{X} so that its restriction to $\pi^{-1}(U)$ is identical to ω via the isomorphism $\pi : \pi^{-1}(U) \simeq U$.*

Proof Let ω be an algebraic m -volume form on U , that is, $\omega \in \Gamma(U, (\wedge^{\text{top}}\Omega_U)^{\otimes m})$. We choose an affine open covering $\{V_\alpha\}_{\alpha \in I}$ of X . Let ω_α be the restriction of ω to

$U \cap V_\alpha$ for each α . Since V_α is affine, by [15, Proposition 5.2(b), Chapter II] we have $\Gamma(V_\alpha, (\mathcal{K}_{V_\alpha})^{\otimes m}) \simeq \Gamma(V_\alpha, \mathcal{K}_{V_\alpha})^{\otimes m}$. For each α we have the following isomorphisms:

$$\begin{aligned} \Gamma(U \cap V_\alpha, \wedge^{\text{top}} \Omega_{U \cap V_\alpha})^{\otimes m} &\simeq \Gamma(V_\alpha, \mathcal{K}_{V_\alpha})^{\otimes m} \\ &\simeq \Gamma(V_\alpha, (\mathcal{K}_{V_\alpha})^{\otimes m}) \\ &\simeq \Gamma(U \cap V_\alpha, (\wedge^{\text{top}} \Omega_{U \cap V_\alpha})^{\otimes m}), \end{aligned}$$

where the the first and the last isomorphisms follow from Lemma 6.27.

Therefore ω_α can be expressed as a finite sum

$$\omega_\alpha = \sum_{i=1}^{n_\alpha} \omega_{\alpha,i,1} \otimes \omega_{\alpha,i,2} \otimes \cdots \otimes \omega_{\alpha,i,m} \quad (n_\alpha \geq 0),$$

where each $\omega_{\alpha,i,k}$ is an algebraic volume form on $U \cap V_\alpha$.

By Lemma 6.24 and Lemma 6.28, the pull-back $\pi^* \omega_{\alpha,i,k}$ is uniquely extended to an algebraic volume form $\tilde{\omega}_{\alpha,i,k}$ on $\pi^{-1}(V_\alpha)$. Put

$$\tilde{\omega}_\alpha := \theta_\alpha \left(\sum_{i=1}^{n_\alpha} \tilde{\omega}_{\alpha,i,1} \otimes \tilde{\omega}_{\alpha,i,2} \otimes \cdots \otimes \tilde{\omega}_{\alpha,i,m} \right),$$

where θ_α is the natural map

$$\theta_\alpha : \Gamma \left(\pi^{-1}(V_\alpha), \wedge^{\text{top}} \Omega_{\pi^{-1}(V_\alpha)} \right)^{\otimes m} \rightarrow \Gamma \left(\pi^{-1}(V_\alpha), (\wedge^{\text{top}} \Omega_{\pi^{-1}(V_\alpha)})^{\otimes m} \right).$$

It is clear that $\tilde{\omega}_\alpha$ is an extension of $\pi^*(\omega_\alpha)$ from $\pi^{-1}(U \cap V_\alpha)$ to $\pi^{-1}(V_\alpha)$.

By Lemma 6.28, for all $\alpha, \beta \in I$, the two algebraic m -volume forms $\tilde{\omega}_\alpha$ and $\tilde{\omega}_\beta$ coincide on $\pi^{-1}(V_\alpha \cap V_\beta)$, since they coincide on $\pi^{-1}(V_\alpha \cap V_\beta \cap U)$. Hence $\{\tilde{\omega}_\alpha\}$ can be glued to be an algebraic m -volume form $\tilde{\omega}$ on \tilde{X} , and clearly the restriction of $\tilde{\omega}$ to $\pi^{-1}(U)$ is identical to ω via the isomorphism $\pi : \pi^{-1}(U) \simeq U$. □

Similar to Proposition 6.25, we have the following proposition.

Proposition 6.30 *Let X be an algebraic variety over F with Gorenstein rational singularities. Let U be a smooth open subvariety of X whose complement has codimension ≥ 2 . Then the measure $(|\omega|_F^{1/m})|^{X(F)}$ is locally finite for all algebraic m -volume form ω on U ($m \geq 1$).*

Proof The proof is similar to that of Proposition 6.25. □

6.8 Locally finiteness of generalized invariant measures

As before, let G be a linear algebraic group over F , with unipotent radical N . Put $G := G(F)$ and $N := N(F)$. Let χ be a character on G which is trivial on N . In this subsection, we prove the following theorem.

Theorem 6.31 *Let X be an algebraic variety over F of Gorenstein rational singularity. Let U be a smooth open subvariety of X whose complement has codimension ≥ 2 . Assume that U is a homogeneous space of G . Let η be a χ -generalized invariant distribution on $U(F)$. Then η is a measure, and $\eta|^{X(F)}$ is locally finite.*

Lemma 6.32 *Theorem 6.31 holds when G is connected.*

Proof We are in the setting of Theorem 6.31 and assume that G is connected. Theorem 6.31 is trivial when $U(F)$ is empty. So assume that $U(F)$ is non-empty. Then we may (and do) assume that $U = G/H$ for some algebraic subgroup H of G . Since $(G/H)(F)$ is the disjoint union of finitely many G -open orbits, we assume without loss of generality that the distribution η is supported on G/H . Write

$$\eta|_{G/H} = P(\text{val} \circ \alpha_1, \text{val} \circ \alpha_2, \dots, \text{val} \circ \alpha_r) \cdot |\beta_1|_F^{s_1} \cdot |\beta_2|_F^{s_2} \cdots \cdots |\beta_t|_F^{s_t} \cdot \chi_f \cdot (|\omega|_F^{1/m})|_{G/H},$$

as in Theorem 6.16. By Lemma 6.27, $\alpha_1, \alpha_2, \dots, \alpha_r, \beta_1, \beta_2, \dots, \beta_t$ extend to elements of $\mathcal{O}(X)$. Therefore

$$P(\text{val} \circ \alpha_1, \text{val} \circ \alpha_2, \dots, \text{val} \circ \alpha_r) \cdot |\beta_1|_F^{s_1} \cdot |\beta_2|_F^{s_2} \cdots \cdots |\beta_t|_F^{s_t}$$

extends to a continuous function on $X(F)$. By Proposition 6.30, $(|\omega|_F^{1/m})|^{X(F)}$ is locally finite. Therefore the lemma follows. □

Denote by G_0 the identity connected component of G .

Lemma 6.33 *Let U be a homogeneous space of G . Then every connected component of U containing an F -point is G_0 -stable and homogeneous.*

Proof Let $x_0 \in U(F)$. Let H denote the stabilizer of x_0 in G . Then $U = G/H$. Note that $G_0 \cdot H$ is an open algebraic subgroup of G , and $(G_0 \cdot H)/H = G_0/(G_0 \cap H)$ is a connected homogeneous space of G_0 . Write

$$G/H = (G_0 \cdot H)/H \sqcup (G \setminus (G_0 \cdot H))/H,$$

and the lemma follows. □

Now we come to the proof of Theorem 6.31 in general. Since X is normal, it is the disjoint union of its irreducible components, and all the irreducible components are open in X . We only need to show that for every irreducible component X' of X ,

$$\eta' := \eta|_{U'(F)} \text{ is a measure, and } \eta'|^{X'(F)} \text{ is locally finite,} \tag{27}$$

where $U' := X' \cap U$. This is trivially true if $U'(F)$ is empty. So assume that $U'(F)$ is non-empty. Then U' is an irreducible component of U , and hence it is also a connected component of U since U is normal. By Lemma 6.33, U' is G_0 -stable and homogeneous. Therefore (27) holds by Lemma 6.32.

6.9 Proof of Theorem 1.5

Now we are in the setting of Theorem 1.5. Let η be a χ -generalized invariant distribution on $\mathbf{U}(\mathbf{F})$. By Theorem 6.31, η is a measure and the measure $\eta|_{\mathbf{X}_f(\mathbf{F})}$ is locally finite. Proposition 6.22 implies that the measure $\eta|_{\mathbf{X}_f(\mathbf{F})}$ is definable of order $\leq k$ for some $k \in \mathbb{N}$. By Corollary 5.21, $\eta|_{\mathbf{X}_f(\mathbf{F})}$ extends to a χ -generalized invariant distribution on $\mathbf{X}(\mathbf{F})$. This finishes the proof of Theorem 1.5.

6.10 A variant of Theorem 1.5

We also have the following theorem.

Theorem 6.34 *Let \mathbf{G} be a linear algebraic group over \mathbf{F} . Let \mathbf{X} be an algebraic variety over \mathbf{F} so that \mathbf{G} acts algebraically on it with an open orbit $\mathbf{U} \subset \mathbf{X}$. Assume that there is a non-zero semi-invariant algebraic volume form on \mathbf{U} , and there is a semi-invariant regular function f on \mathbf{X} with the following properties:*

- f does not vanish on \mathbf{U} , and $\mathbf{X}_f \setminus \mathbf{U}$ has codimension ≥ 2 in \mathbf{X}_f , where \mathbf{X}_f denotes the complement in \mathbf{X} of the zero locus of f ;
- the variety \mathbf{X}_f has rational singularities.

Let χ be a character of $\mathbf{G}(\mathbf{F})$ which is trivial on $\mathbf{N}(\mathbf{F})$, where \mathbf{N} denotes the unipotent radical of \mathbf{G} . Then every generalized χ -invariant distribution on $\mathbf{U}(\mathbf{F})$ extends to a generalized χ -invariant distribution on $\mathbf{X}(\mathbf{F})$.

The proof of Theorem 6.34 is the same as that of Theorem 1.5, except that we should replace Theorem 6.31 by the following theorem.

Theorem 6.35 *Let \mathbf{X} be an algebraic variety over \mathbf{F} of rational singularities. Let \mathbf{U} be a smooth open subvariety of \mathbf{X} whose complement has codimension ≥ 2 . Assume that \mathbf{U} is a homogeneous space of \mathbf{G} and there exists a non-zero semi-invariant algebraic volume form on \mathbf{U} . Let η be a χ -generalized invariant distribution on $\mathbf{U}(\mathbf{F})$, where χ is as in Theorem 6.34. Then η is a measure, and $\eta|_{\mathbf{X}(\mathbf{F})}$ is locally finite.*

The proof of Theorem 6.35 is also similar to that of Theorem 6.31, except that we replace Proposition 6.30 by Proposition 6.25.

7 Generalized semi-invariant distributions on matrix spaces

We consider the following action of $G := \mathrm{GL}_m(\mathbf{F}) \times \mathrm{GL}_n(\mathbf{F})$ ($m, n \geq 1$) on the space $\mathbf{M}_{m,n} := \mathbf{M}_{m,n}(\mathbf{F})$ of $m \times n$ -matrices with coefficients in \mathbf{F} :

$$(g_1, g_2) \cdot x := g_1 x g_2^{-1}, \quad g_1 \in \mathrm{GL}_m(\mathbf{F}), \quad g_2 \in \mathrm{GL}_n(\mathbf{F}), \quad x \in \mathbf{M}_{m,n}.$$

For $r = 0, 1, \dots, \min\{m, n\}$, let \mathbf{O}_r denote the set of rank r matrices in $\mathbf{M}_{m,n}$, which is a G -orbit. Put $\bar{\mathbf{O}}_r := \bigsqcup_{i=0}^r \mathbf{O}_i$. Then \mathbf{O}_r is open and dense in $\bar{\mathbf{O}}_r$. Every character of G is given by

$$(g_1, g_2) \mapsto \chi_1(\det(g_1))\chi_2(\det(g_2)),$$

for some characters χ_1, χ_2 of F^\times . We denote this character of G by the pair (χ_1, χ_2) . Let $I_r = (a_{ij})$ be the matrix in O_r such that

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leq i = j \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

The stabilizer group G_r of I_r in G consists of elements of the form

$$\left(\begin{bmatrix} x & y \\ 0 & w_1 \end{bmatrix}, \begin{bmatrix} x & 0 \\ z & w_2 \end{bmatrix} \right),$$

where $x \in GL_r(F), y \in M_{r,m-r}(F), z \in M_{n-r,r}(F), w_1 \in GL_{m-r}(F), w_2 \in GL_{n-r}(F)$. The algebraic modular character Δ_{G_r} of G_r is given by

$$\left(\begin{bmatrix} x & y \\ 0 & w_1 \end{bmatrix}, \begin{bmatrix} x & 0 \\ z & w_2 \end{bmatrix} \right) \mapsto \det(x)^{m-n} \det(w_1)^{-r} \det(w_2)^r.$$

By (16) and (25), for each character $\underline{\chi}$ of G , the orbit O_r is $\underline{\chi}$ -admissible if and only if

$$\underline{\chi}|_{G_r} = |\Delta_G|_F \cdot |\Delta_{G_r}|_F^{-1} = |\Delta_{G_r}|_F^{-1}.$$

Since the stabilizer G_r is connected when viewed as an algebraic group, the orbit O_r is $\underline{\chi}$ -admissible if and only if it is weakly $\underline{\chi}$ -admissible.

Proposition 7.1 *Fix a character $\underline{\chi} = (\chi_1, \chi_2)$ of G . Assume that $m \neq n$, then the following holds.*

- (a) If $\underline{\chi} = (1, 1)$, then $D(M_{m,n})^{\underline{\chi}, \infty} = D(M_{m,n})^{\underline{\chi}} = \mathbb{C} \cdot \delta_0$, where δ_0 is the delta distribution supported at 0.
- (b) If $\underline{\chi} = (|\cdot|_F^n, |\cdot|_F^{-m})$, then $D(M_{m,n})^{\underline{\chi}, \infty} = D(M_{m,n})^{\underline{\chi}} = \mathbb{C} \cdot \mu_{M_{m,n}}$, where $\mu_{M_{m,n}}$ is a Haar measure on $M_{m,n}$.
- (c) If $\underline{\chi} \neq (1, 1), (|\cdot|_F^n, |\cdot|_F^{-m})$, then $D(M_{m,n})^{\underline{\chi}, \infty} = 0$.

Proof Note that there is no non-constant semi-invariant regular function on each orbit O_r . By Theorem 6.16, all generalized $\underline{\chi}$ -invariant distributions on O_r are $\underline{\chi}$ -invariant.

If $\underline{\chi} = (1, 1)$, then O_0 is the only $\underline{\chi}$ -admissible orbit, and therefore

$$D(M_{m,n})^{\underline{\chi}, \infty} = D(M_{m,n})^{\underline{\chi}} = \mathbb{C} \cdot \delta_0.$$

If $\underline{\chi} = (|\cdot|_F^n, |\cdot|_F^{-m})$, then $O_{\min\{m,n\}}$ is the only $\underline{\chi}$ -admissible orbit. Then Theorem 1.4 implies that

$$\begin{aligned} \mathbb{C} \cdot \mu_{M_{m,n}} &\subset D(M_{m,n})^{\underline{\chi}} \subset D(M_{m,n})^{\underline{\chi}, \infty} \\ &= D(O_{\min\{m,n\}})^{\underline{\chi}, \infty} = D(O_{\min\{m,n\}})^{\underline{\chi}} = \mathbb{C} \cdot \mu_{M_{m,n}}. \end{aligned}$$

If $\underline{\chi} \neq (1, 1), (|\cdot|_F^n, |\cdot|_F^{-m})$, then each orbit O_r is not $\underline{\chi}$ -admissible. Therefore $D(M_{m,n})^{\underline{\chi}, \infty} = 0$. □

We now assume that $m = n$, and we denote by M_n the space $M_{n,n}$. Consider the following zeta integral

$$Z_\chi(\phi, s) = \int_{M_n} \phi(x)\chi(\det(x))|\det(x)|_{\mathbb{F}}^s \frac{dx}{|\det(x)|_{\mathbb{F}}^n},$$

where \det is the determinant function on M_n , dx is the Haar measure on M_n so that the space $M_n(\mathbb{R})$ of integral matrices in M_n has volume 1, χ is a character of \mathbb{F}^\times and $\phi \in \mathcal{S}(M_n)$. By Theorem 5.13, it is a rational function of $1 - q_{\mathbb{F}}^{-s}$. Let $Z_{\chi,i}$ be the i -th coefficient of the Laurent expansion of Z_χ (as a rational function of $1 - q_{\mathbb{F}}^{-s}$). By Proposition 5.20, $Z_{\chi,i}$ is a generalized (χ, χ^{-1}) -invariant distribution. It is easy to check that

$$(1 - g) \cdot Z_{\chi,i} = Z_{\chi,i-1}, \tag{28}$$

for all $g = (g_1, g_2) \in G$ such that $\det(g_1^{-1}g_2)$ is a uniformizer of \mathbb{R} . Here the action of G on $D(M_n)^{(\chi, \chi^{-1}), \infty} \subset \text{Hom}_{\mathbb{C}}(\mathcal{S}(M_n), (\chi, \chi^{-1}))$ is as in the Eq. (5) of Sect. 2.2.

- Proposition 7.2** (a) *If $\chi = |\cdot|_{\mathbb{F}}^r$ for some $r = 0, 1, \dots, n - 1$, then $Z_{\chi,i} = 0$ for all $i < -1$, and $\{Z_{\chi,i}\}_{i \geq -1}$ is a basis of $D(M_n)^{(\chi, \chi^{-1}), \infty}$.*
 (b) *If $\chi \neq |\cdot|_{\mathbb{F}}^r$ for all $r = 0, 1, \dots, n - 1$, then $Z_{\chi,i} = 0$ for all $i < 0$, and $\{Z_{\chi,i}\}_{i \geq 0}$ is a basis of $D(M_n)^{(\chi, \chi^{-1}), \infty}$.*
 (c) *For every character (χ_1, χ_2) of G , the space $D(M_n)^{(\chi_1, \chi_2), \infty} = 0$ if $\chi_1\chi_2 \neq 1$.*

Proof Note that for each $i < 0$, $Z_{\chi,i}$ is supported in \bar{O}_{n-1} , in other words,

$$Z_{\chi,i} \in D(\bar{O}_{n-1})^{(\chi, \chi^{-1}), \infty}.$$

It is easy to see $\{Z_{\chi,i}|_{O_n}\}_{i \geq 0}$ is a basis of $D(O_n)^{(\chi, \chi^{-1}), \infty}$. In particular, we have an exact sequence

$$0 \rightarrow D(\bar{O}_{n-1})^{(\chi, \chi^{-1}), \infty} \rightarrow D(M_n)^{(\chi, \chi^{-1}), \infty} \rightarrow D(O_n)^{(\chi, \chi^{-1}), \infty} \rightarrow 0. \tag{29}$$

If $\chi \neq |\cdot|_{\mathbb{F}}^r$ for all $r = 0, 1, \dots, n - 1$, then any orbit in \bar{O}_{n-1} is not χ -admissible, i.e. not weakly χ -admissible. By Bernstein-Zelevinsky localization principle,

$$D(\bar{O}_{n-1})^{(\chi, \chi^{-1}), \infty} = 0.$$

Therefore part (b) of the proposition follows.

Now assume that $\chi = |\cdot|_{\mathbb{F}}^r$ ($r = 0, 1, \dots, n - 1$). Then

$$D(\bar{O}_{n-1})^{(\chi, \chi^{-1}), \infty} = D(\bar{O}_r)^{(\chi, \chi^{-1}), \infty} = D(O_r)^{(\chi, \chi^{-1}), \infty} = D(O_r)^{(\chi, \chi^{-1})}. \tag{30}$$

Here the first equality follows from the localization principle of Bernstein-Zelevinsky, the second one is implied by Theorem 1.4, and the last one follows as in the proof

of Proposition 7.1. In particular, $Z_{\chi,i}$ is (χ, χ^{-1}) -invariant for all $i < 0$. Then (28) implies that $Z_{\chi,i} = 0$ for all $i < -1$. On the other hand, the computation ([18, Chapter 10.1])

$$\int_{M_n(\mathbb{R})} |\det(x)|^s dx = \prod_{i=1}^n \frac{1 - q_F^{-i}}{1 - q_F^{-i-s}}$$

implies that $Z_{\chi,-1} \neq 0$. Therefore $Z_{\chi,-1}$ is a generator of the one-dimensional space (30). Now part (a) of the proposition follows by the exact sequence (29).

Part (c) of the proposition is an easy consequence of Bernstein-Zelevinsky localization principle, since under which condition every orbit in M_n is not χ -admissible. \square

In view of (28), Proposition 7.2 implies that

$$\dim D(M_n)^{(\chi, \chi^{-1})} = 1$$

for all character χ of F^\times . This generalizes the equality (2) of Tate's thesis, and is a (well-known) particular case of local theta correspondence.

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