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# MIRKOVIĆ-VILONEN CYCLES AND POLYTOPES FOR A SYMMETRIC PAIR

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Abstract. Let $G$ be a connected, simply-connected, and almost simple algebraic group, and let $\sigma$ be a Dynkin automorphism on $G$. Then $(G, G^\sigma)$ is a symmetric pair. In this paper, we get a bijection between the set of $\sigma$-invariant MV cycles (polytopes) for $G$ and the set of MV cycles (polytopes) for $G^\sigma$, which is the fixed point subgroup of $G$; moreover, this bijection can be restricted to the set of MV cycles (polytopes) in irreducible representations. As an application, we obtain a new proof of the twining character formula.

## 1. Introduction

Let $G$ be a connected semisimple algebraic group over $\mathbb{C}$, and let $\mathcal{G}$ be the affine Grassmannian of $G$. Let $\mathcal{G}_\lambda$ be the $G(\mathbb{C}[[t]])$-orbit on $\mathcal{G}$ corresponding to a dominant coweight $\lambda$ on $G$. Let $IC_\lambda$ be the spherical perverse sheaf supported on $\overline{\mathcal{G}_\lambda}$. V. Ginzburg [G] and Mirković and Vilonen [MV] set up the geometric Satake correspondence, which says that the category of spherical perverse sheaves on $\mathcal{G}$ is equivalent to the category of finite dimensional representations of the Langlands dual group $G^\vee$ of $G$; in particular, the irreducible representation $V(\lambda)$ of $G^\vee$ with highest weight $\lambda$ is identified with the cohomology group $H^*(\mathcal{G}, IC_\lambda)$. Furthermore, Mirković and Vilonen [MV] discovered Mirković-Vilonen cycles which affords a natural basis of $V(\lambda)$.

In [A], Anderson studied the moment polytopes of Mirković-Vilonen cycles, which are called Mirković-Vilonen polytopes, and showed that these polytopes could be used to understand the combinatorics of representations of $G^\vee$. In [K1], Kamnitzer gave an explicit combinatorial description of the MV cycles and polytopes. He showed that canonical basis and MV cycles are governed by the same combinatorics, i.e., MV cycles $\longleftrightarrow$ MV polytopes $\longleftrightarrow$ canonical basis, are bijections.

Let $\sigma$ be a nontrivial Dynkin automorphism of $G$. We have a Dynkin automorphism on $G^\vee$ induced from $\sigma$. Let $G^\sigma$ be the identity component of a fixed point group of $\sigma$ on $G$. Let $\lambda$ be a $\sigma$-invariant dominant coweight of $G$, which can also be viewed as a dominant coweight of $G^\sigma$. Let $v(\lambda)$ be the irreducible representation of $G^\vee$ with highest weight $\lambda$. We have a natural action of $\sigma$ on $V(\lambda)$ induced from the action of the automorphism on $G^\vee$, which fixes the highest weight vector in $V(\lambda)$. For a $\sigma$-invariant coweight $\mu$ for $G$, $\sigma$ acts on the weight space $V_\mu(\lambda)$. The twining character $\mathrm{ch}^\sigma V(\lambda)$ is defined to be $\sum_{\sigma(\mu)=\mu} \mathrm{trace}(\sigma|_{V_\mu(\lambda)}) e^\mu$. It is related to the character of the irreducible representation of $(G^\sigma)^\vee$ with highest weight $\lambda$ through the twining character formula, which is attributed to Jantzen [J] under the name

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of Jantzen theorem in [KLP]. Though there are many proofs in the literature (for example, [J], [N], [KLP]), it seems that there is no satisfactory explanation for why Langlands dual appears in this formula.

In this paper, we consider the action of $\sigma$ on MV cycles and MV polytopes. The main result of the paper is to give an explicit bijection between $\sigma$-invariant MV cycles (polytopes) for $G$ to MV cycles (polytopes) for $G^\sigma$. In terms of polytopes, it sends $\sigma$-invariant MV polytopes $P$ for $G$, to $P^\sigma$, which is a MV polytope for $G^\sigma$. The bijection can be restricted to MV cyles (polytopes) in irreducible representation space.

In this paper, we also show that the automorphism on $G^\vee$ from Tannakian formalism is a Dynkin automorphism. On $V(\lambda)$, there are two actions of $\sigma$, where one is induced from $G^\vee$, and the other one is induced from the action of $\sigma$ on MV cycles. We show that both of them agree, then we get a new proof of twining character formula through geometric Satake correspondence.

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## 2. Dynkin automorphism

2.1. **Notations.** Let $G$ be a connected, simply-connected and almost simple algebraic group of rank $\ell$ over $\mathbb{C}$. Let $T$ be a maximal torus of $G$ and let $X^* = \mathrm{Hom}(T, \mathbb{C}^\times)$, $X_* = \mathrm{Hom}(\mathbb{C}^\times, T)$ denote the weight and coweight lattices of $T$. Then we have a natural perfect pairing $\langle , \rangle : X_* \times X^* \to \mathbb{Z}$. Let $W = N(T)/T$ denote the Weyl group.

Let $I = \{1, \cdots, l\}$ denote vertices of the Dynkin diagram of $G$. Let $B$ be a Borel subgroup of $G$ containing $T$. Let $\alpha_1, \alpha_2, \cdots, \alpha_l$ and $\alpha_1^\vee, \alpha_2^\vee, \cdots, \alpha_l^\vee$ be simple roots and simple coroots of $G$ with respect to $B$, respectively. Then $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ is the entry of the Cartan matrix of $G$. Note that $(X_*, X^*, \langle , \rangle, \alpha_i^\vee, \alpha_i; i \in I)$ is the root datum of $G$. Let $\lambda_1, \cdots, \lambda_l \in X^* \otimes \mathbb{R}$ be fundamental weights.

For $i \in I$, let $x_i : \mathbb{C} \to G$ and $y_i : \mathbb{C} \to G$ be root homomorphisms (corresponding to $\alpha_i$ and $-\alpha_i$, respectively) which together with $T$, $B$ form a pinning of $G$.

Let $s_1, \cdots, s_\ell \in W$ be the set of simple reflections. Let $w_0$ be the longest element of $W$, and let $m$ be its length.

We use $\geq$ for the usual partial order on $X_*$, so that $\mu \geq \nu$ if and only if $\mu - \nu$ is a sum of positive coroots. More generally, for each $w \in W$, we have the twisted partial order $\geq_w$ on $X_*$, where $\mu \geq_w \nu$ if and only if $w^{-1} \cdot \mu \geq w^{-1} \cdot \nu$.

A reduced word for an element $w \in W$ is a sequence of indices $\mathbf{i} = (i_1, \cdots, i_k) \in I^k$ such that $w = s_{i_1} \cdot s_{i_2} \cdots s_{i_k}$ is a reduced expression. In this paper, a reduced word will always mean a reduced word for $w_0$, where $w_0$ is the longest element in $W$.

2.2. **Group structure of $G^\sigma$.** Let $\sigma : I \to I$ be a nontrivial bijection, satisfying $a_{\sigma(i)\sigma(j)} = a_{ij}$ for all $i, j \in I$. We assume that there are automorphisms $\sigma : X^* \to X^*$ and $\sigma : X_* \to X_*$ of **Z**-modules satisfying $\sigma(\alpha_i) = \alpha_{\sigma(i)}$ and $\sigma(\alpha_i^\vee) = \alpha_{\sigma(i)}^\vee$ for any $i \in I$. Then $\sigma$ induces an automorphism $\sigma : G \to G$ of algebraic groups, such that $\sigma(x_i(a)) = x_{\sigma(i)}(a)$ and $\sigma(y_i(a)) = y_{\sigma(i)}(a)$ ($\forall\ i \in I$). We call $\sigma$ a Dynkin automorphism on $G$. In particular, we have $\sigma(B) = B$ and $\sigma(T) = T$.

Let $G^\sigma$ be the fixed point group of $\sigma$ on $G$, and let $T^\sigma$ and $B^\sigma$ be the fixed point groups of $T$ and $B$, respectively. Then $G^\sigma$, $B^\sigma$ and $T^\sigma$ are connected; moreover, $G^\sigma$ is almost simple algebraic group, under our assumptions on $G$ (see [ST]). We call $(G, G^\sigma)$ a symmetric pair.

We set $X_*^\sigma = \{\lambda \in X_* | \sigma(\lambda) = \lambda\}$, and $X_\sigma^* = \mathrm{Hom}(X_*^\sigma, \mathbb{Z})$. We have a perfect pairing $X_*^\sigma \times X_\sigma^* \to \mathbb{Z}$ denoted again by $\langle , \rangle$. Let $I_\sigma$ be the set of $\sigma$-orbits on $I$.

For any $\eta \in I_\sigma$, let $\alpha_\eta^\vee = 2^h \sum_{i\in\eta} \alpha_i^\vee \in X_*^\sigma$, where $h$ is the number of unordered pairs $(i, j)$ such that $i, j \in \eta$, $\alpha_i + \alpha_j \in \Phi$. Note that $h = 1$, if $\eta = \{i, j\}$ and $a_{ij} = -1$; $h = 0$, otherwise. Let $\theta : X^* \otimes \mathbb{R} \to X_\sigma^* \otimes \mathbb{R}$ be the natural surjection induced from the perfect pairing $\langle , \rangle : X_* \times X^* \to \mathbb{Z}$. Set $\alpha_\eta = \theta(\alpha_i)$, and $\lambda_\eta = \frac{1}{h}\theta(\lambda_i)$, where $i$ is any element of $\eta$. We have the following proposition (see [KLP], [J]).

**Proposition 2.1.** *$(X_*^\sigma, X_\sigma^*, \alpha_\eta^\vee, \alpha_\eta)$ is a root datum of $G^\sigma$.*

Define $x_\eta = \prod_{i\in\eta} x_i : \mathbb{C} \to G^\sigma$, by $x_\eta(a) = \prod_{i\in\eta} x_i(a)$, if $\eta$ has only one element, or $\forall\ i, j \in \eta$, with $i \neq j$, $a_{ij} = 0$; define $x_\eta : \mathbb{C} \to G^\sigma$, by $x_\eta(a) = x_i(a)x_j(2a)x_i(a)$, if $\eta = \{i, j\}$, $a_{ij} = -1$. We have the following lemma (see [L1]).

**Lemma 2.2.** *Let $x_1$, $x_2$ be two simple root subgroup homomorphisms of $G$ of type $A_2$ corresponding to $\alpha_1$ and $\alpha_2$. Then we have*

$$x_1(a_1)x_2(a_2)x_1(a_3) = x_2(\frac{a_2a_3}{a_1 + a_3})x_1(a_1 + a_3)x_2(\frac{a_1a_2}{a_1 + a_3}).$$

From this lemma, we see easily that $x_\eta$ is a group homomorphism. Similarly, we can define $y_\eta$, so that $x_\eta$ and $y_\eta$ are homomorphisms from $\mathbb{C}$ to $G^\sigma$. Since $tx_\eta(a)t^{-1} = x_\eta(\alpha_\eta(t)a)$, $x_\eta$ is a root subgroup homomorphism of $G^\sigma$ with root $\alpha_\eta$. We have

**Proposition 2.3.** *$(T^\sigma, B^\sigma, x_\eta, y_\eta; \eta \in I_\sigma)$ form a pinning of $G^\sigma$.*

Clearly, $\sigma : G \to G$ induces an automorphism of $W$ denoted again by $\sigma$, satisfying $\sigma(s_i) = s_{\sigma(i)}$ for any $i \in I$. Let $W^\sigma = \{w \in W | \sigma(w) = w\}$. For any $\eta \in I_\sigma$ we define $s_\eta \in W^\sigma$ to be the longest element in the subgroup of $W$ generated by $\{s_i; i \in \eta\}$. It is known that $W^\sigma$ is a Coxeter group on the generators $\{s_\eta; \eta \in I_\sigma\}$. Any element $w \in W^\sigma$ can be restricted to $X_*^\sigma$. Under this restriction, we can see that $W^\sigma$ is identified with the Weyl group of $G^\sigma$. For $w \in W^\sigma$, we denote by $\ell_\sigma(w)$ the length of $w$ in the Coxeter group $W^\sigma$.

## 3. MV cycles and MV polytopes for the symmetric pair

3.1. **Action of $\sigma$ on affine Grassmannian.** Let $\mathcal{O} = \mathbb{C}[[t]]$, and let $\mathcal{K}$ be the quotient field of $\mathcal{O}$. Let $\mathcal{G}$ and $\mathcal{G}_\sigma$ be affine Grassmannian of $G$ and $G^\sigma$, respectively. As the sets of rational points over $\mathbb{C}$, $\mathcal{G} = G(\mathcal{K})/G(\mathcal{O})$, and $\mathcal{G}_\sigma = G(\mathcal{K})^\sigma/G(\mathcal{O})^\sigma$. A coweight $\mu \in X_*$ gives a point in $\mathcal{G}$, denoted by $\underline{t}^\mu$. It is known that $\underline{t}^\mu$ is a fixed point for the action of $T$ on $\mathcal{G}$. In fact, all the fixed points of $T$ are given in this way.

For a given dominant coweight $\lambda$, we set $\mathcal{G}^\lambda = G(\mathcal{O})\cdot \underline{t}^\lambda$. We have the decomposition $\mathcal{G} = \bigsqcup_{\lambda\in X_*^+}\mathcal{G}^\lambda$, where $X_*^+$ is the set of dominant coweights.

Let $N$ be the unipotent radical of $B$. For $w\in W$, we set $N_w = wNw^{-1}$. For $w\in W$ and $\mu\in X_*$, define the semi-infinite cells by $S_w^\mu = N_w(\mathcal{K})\cdot \underline{t}^\mu$. For simplicity, we set $S^\mu = S_e^\mu = N(\mathcal{K})\cdot \underline{t}^\mu$. We have $\mathcal{G} = \bigsqcup_{\mu\in X_*} S^\mu$. The semi-infinite cells have the simple containment relation, $\overline{S_w^\mu} = \bigsqcup_{\nu\leq_w \mu} S_w^\nu$. We see that if $S_w^\mu \cap S_v^\nu \neq \varnothing$, then $\nu\leq_w \mu$.

We have the closed embedding $\iota : \mathcal{G}_\sigma \hookrightarrow \mathcal{G}$. Since $\sigma(S^\lambda) = S^{\sigma(\lambda)}$, we have $\mathcal{G}^\sigma = \bigsqcup_{\lambda\in X_*^\sigma}(S^\lambda)^\sigma$.

Set $U := \{g(t^{-1}) \in G(\mathbb{C}[t^{-1}])|g(0) = 1)\}$. Then the fixed point set $U^\sigma = \{g(t^{-1}) \in G^\sigma(\mathbb{C}[t^{-1}])|g(0) = 1)\}$. For a coweight $\lambda$, set $S(\lambda) := N(\mathbb{C}[t,t^{-1}])\cap t^\lambda U t^{-\lambda}$ and $S_\sigma(\lambda) := N^\sigma(\mathbb{C}[t,t^{-1}])\cap t^\lambda U^\sigma t^{-\lambda}$.

The following result is well known.

**Lemma 3.1.** *Let $\lambda\in X_*$. Then the group $S(\lambda)$ acts simply-transitively on $S^\lambda$, i.e., $S(\lambda)\simeq S^\lambda$, with the map $g\mapsto g.t^\lambda$.*

**Proposition 3.2.** *The fixed point subvariety of the action of $\sigma$ on $\mathcal{G}$ is exactly identified with $\mathcal{G}_\sigma$.*

*Proof.* From Lemma 3.1, we are reduced to showing that $S(\lambda)^\sigma = S_\sigma(\lambda)$ for $\lambda\in X_*^\sigma$, and it is easy to see, since

$$S(\lambda)^\sigma = N(\mathbb{C}[t,t^{-1}])^\sigma \cap (t^\lambda U t^{-\lambda})^\sigma = N^\sigma(\mathbb{C}[t,t^{-1}])\cap t^\lambda U^\sigma t^{-\lambda} = S_\sigma(\lambda). \qquad \square$$

From $\overline{\mathcal{G}^\lambda} = \bigsqcup_{\mu\leq\lambda}\mathcal{G}^\mu$, $\overline{S_w^\mu} = \bigsqcup_{\nu\leq_w\mu} S_w^\nu$ and the above proposition, we can easily see that

**Corollary 3.3.** *For $\lambda$ a $\sigma$-invariant, and $w$ a $\sigma$-invariant element in $W$, we have $(\mathcal{G}^\lambda)^\sigma = \mathcal{G}_\sigma^\lambda$, $\overline{\mathcal{G}^\lambda}^\sigma = \overline{\mathcal{G}_\sigma^\lambda}$, $(S_w^\mu)^\sigma = (S_\sigma)_w^\mu$, and $\overline{S_w^\mu}^\sigma = \overline{(S_\sigma)_w^\mu}$.*

3.2. **MV cycles and MV polytopes.** Let $\mu_1$, $\mu_2$ be coweights with $\mu_1\geq\mu_2$. Following Anderson [A], an irreducible component of $\overline{S_e^{\mu_1}\cap S_{w_0}^{\mu_2}}$ is called an MV cycle with coweight $(\mu_1,\mu_2)$. This definition of an MV cycle is a generalization of the original one in [MV]. $X_*$ acts on $\mathcal{G}$ by $\nu\cdot L := t^\nu\cdot L$. Since $T$ normalizes $N_w$, we see that $\nu\cdot S_w^\mu = S_w^{\mu+\nu}$. If $A$ is a component of $\overline{S_e^{\mu_1}\cap S_{w_0}^{\mu_2}}$, then $\nu\cdot A$ is a component of $\overline{S_e^{\mu_1+\nu}\cap S_{w_0}^{\mu_2+\nu}}$. Hence $X_*$ acts on the set of all MV cycles. The orbit of an MV cycle with coweight $(\mu_1,\mu_2)$ is called a stable MV cycle with coweight $\mu_2-\mu_1$. Note that a stable MV cycle with coweight $\mu$ has a unique representative with coweight $(\nu,\nu+\mu)$ for a fixed coweight $\nu$.

Let $\mathrm{MVC}_G$ denote the set of stable MV cycles for $G$, and let $\mathrm{MVC}_G^\mu$ denote the set of those with coweight $\mu$. For a $T$-invariant closed subvariety $A$ of the affine Grassmannian, let $\Phi(A)\subset t_{\mathbb{R}} := X_*\otimes\mathbb{R}$ be the moment polytope of $A$, which is exactly the convex hull of $\{\mu\in X_*|t^\mu\in A\}$.

If $A$ is an MV cycle with coweight $(\mu_1,\mu_2)$, then we say that $\Phi(A)$ is an MV polytope with coweight $(\mu_1,\mu_2)$. The action of $X_*$ on the set of MV cycles gives an action of $X_*$ on the set of MV polytopes. It is easy to see that $\nu\cdot P = P+\nu$. The orbit of $X_*$ on an MV polytope with coweight $(\mu_1,\mu_2)$ is called a stable MV polytope with coweight $\mu_2-\mu_1$.

Let $\mathrm{MVP}_G$ be the set of stable MV polytopes for $G$, and let $\mathrm{MVP}_G^\mu$ be the set of stable MV polytopes for $G$ with coweight $\mu$. As mentioned in [A], there is a natural bijection between $\mathrm{MVC}_G$ and $\mathrm{MVP}_G$. Let $C$ be an MV cycle, and let $[C]$ be its stable MV cycle. Let $P_C$ be the corresponding MV polytope of $C$, and let $[P_C]$ be its stable MV polytope. If there is no confusion, we write $C$ (resp. $P$) for both MV cycle (or polytope) and stable MV cycle (resp. polytope).

Suppose we are given a collection of coweights $\mu_\bullet = (\mu_w)_{w\in W}$ such that $\mu_v \leq_w \mu_w$ for all $v, w \in W$. Then we define the corresponding pseudo-Weyl polytope by

$$P(\mu_\bullet) := \bigcap_w C_w^{\mu_w} = \{\alpha | \langle \alpha, w\cdot\lambda_i\rangle \leq \langle \mu_w, w\cdot\lambda\rangle,\ \forall\, w\in W, \text{ and } i \in I\}.$$

For a collection $(\mu_w)_{w\in W}$ with coweights such that $\mu_y \leq_w \mu_w$, for any $y, w \in W$, set $A(\mu_\bullet) = \bigcap S_w^{\mu_w}$, and let $\mathrm{Conv}(\mu_\bullet)$ be the convex hull of $(\mu_w)_{w\in W}$ in $t_{\mathbb{R}}$. $A(\mu_\bullet)$ is called a *GGMS stratum*, and it is a candidate of MV cycles. If it is not empty, then the moment polytope of $\overline{A(\mu_\bullet)}$ is exactly $\mathrm{Conv}(\mu_\bullet)$ (see Lemma 2.3, [K1]), which also coincides with $P(\mu_\bullet)$. That is, $\mathrm{Conv}(\mu_\bullet) = P(\mu_\bullet)$.

The following theorem gives a criterion for the closure of a GGMS stratum to be an MV cycle.

**Theorem 1** (Kamnitzer[K1]). *Let $(\mu_w)_{w\in W}$ be the set with coweights, such that $\mu_y \leq_w \mu_w$, for any $y, w \in W$. Then $\overline{A(\mu_\bullet)} = \overline{\bigcap S_w^{\mu_w}}$ is an MV cycle if and only if* $\mathrm{Conv}(\mu_\bullet)$ *is an MV polytope.*

Let $P$ be an MV polytope with vertices $(\mu_w)_{w\in W}$. Then $P$ is the moment polytope of an MV cycle $\overline{\bigcap S_w^{\mu_w}}$. In this case, $\sigma(\overline{\bigcap S_w^{\mu_w}}) = \overline{\bigcap S_w^{\sigma(\mu_{\sigma^{-1}(w)})}}$ is also an MV cycle, and its moment polytope is exactly $\mathrm{Conv}(\sigma(\mu_{\sigma^{-1}(w)}))$. Hence it is an MV polytope with vertices $(\sigma(\mu_{\sigma^{-1}(w)}))_{w\in W}$, which coincides with $\sigma(P)$.

**Lemma 3.4.** *Let $(\mu_w)_{w\in W}$ be the vertices of an MV polytope $P$, and let $A(\mu_\bullet)$ be the corresponding GGMS stratum, such that $\overline{A(\mu_\bullet)}$ is an MV cycle. Then the following statements are equivalent:*

(1) *$P$ is $\sigma$-invariant.*
(2) *$\overline{A(\mu_\bullet)}$ is $\sigma$-invariant.*
(3) *$A(\mu_\bullet)$ is $\sigma$-invariant.*
(4) *$\sigma(\mu_w) = \mu_{\sigma(w)}$, $\forall\ w \in W$.*

*Proof.* Since MV cycles are parametrized by MV polytopes bijectively, it is easy to see that the moment polytope of $\sigma(\overline{\bigcap S_w^{\mu_w}})$ is $\sigma(P)$. So $P$ is $\sigma$-invariant if and only if $\overline{A(\mu_\bullet)}$ is $\sigma$-invariant, i.e., (1) ⇔ (2).

Assume $\overline{A(\mu_\bullet)}$ is $\sigma$-invariant. Then $\overline{\bigcap S_w^{\mu_w}} = \overline{\bigcap S_w^{\sigma(\mu_{\sigma^{-1}(w)})}}$. Since $\bigcap S_w^{\mu_w}$ and $\bigcap S_w^{\sigma(\mu_{\sigma^{-1}(w)})}$ are locally closed, we have $(\bigcap S_w^{\mu_w}) \cap (\bigcap S_w^{\sigma(\mu_{\sigma^{-1}(w)})}) \neq \varnothing$. It implies that, $\forall\ w \in W$, $S_w^{\mu_w} \cap S_w^{\sigma(\mu_{\sigma^{-1}(w)})} \neq \varnothing$. Hence $\mu_w = \sigma(\mu_{\sigma^{-1}(w)})$, $\forall w \in W$. So (2) ⇒ (4).

It is easy to see (3) ⇔ (4), and (4) implies (1) immediately. □

3.3. **Lusztig datum.** Let $\mathbf{i}$ be a reduced word, and $n_\bullet \in \mathbb{N}^m$. Recall some results in [K1]. We define $\{\mu_{w_k^{\mathbf{i}}}\}_{0\leq k\leq m}$ inductively by $\mu_e = 0$ and $\mu_{w_k^{\mathbf{i}}} = \mu_{w_{k-1}^{\mathbf{i}}} - n_k w_{k-1}^{\mathbf{i}}(\alpha_{i_k}^\vee)$, for any $1 \leq k \leq m$. Set $A^{\mathbf{i}}(n_\bullet) = \bigcap S_{w_k^{\mathbf{i}}}^{\mu_{w_k^{\mathbf{i}}}}$. Then $\overline{A^{\mathbf{i}}(n_\bullet)}$ is an MV cycle with coweight $\mu_{w_0}$, and the corresponding MV polytope $P$ has $\mathbf{i}$-Lusztig datum $n_\bullet$.

From the corresponding **i**-Lusztig datum of the MV polytope $P$, we can recover the vertices of $P$ uniquely, through the above procedure. In this way, we have a bijection from MV polytopes to **i**-Lusztig data. Moreover, there exists an explicit bijection between **i**-Lusztig data and MV cycles, $\tau_{\mathbf{i}}:\mathbb{N}^m\to \mathrm{MVC}$ by $\tau_{\mathbf{i}}(n_\bullet)=\overline{A^{\mathbf{i}}(n_\bullet)}$.

Let $\mathbf{i}$, $\mathbf{i}'$ be two reduced words of $w_0$. It is known that $\mathbf{i}'$ can be obtained from $\mathbf{i}$ through several braid moves. Fix a path of braid moves from $\mathbf{i}$ to $\mathbf{i}'$. For each move, there is a transform (in Proposition 5.2, [K1]) between the Lusztig data of $P$ along the two consecutive reduced words. By combining these transforms, we get a bijection $R_{\mathbf{i}}^{\mathbf{i}'}:\mathbb{N}^m\to\mathbb{N}^m$, which is independent of the choice of the path from $\mathbf{i}$ to $\mathbf{i}'$. We call it the Lusztig transform from $\mathbf{i}$ to $\mathbf{i}'$ for $G$. From [K1], we also know that $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet)=n'_\bullet$ if and only if $A^{\mathbf{i}}(n_\bullet)\cap A^{\mathbf{i}'}(n'_\bullet)$ is dense in $A^{\mathbf{i}}(n_\bullet)$.

We give a necessary and sufficient condition on the **i**-Lusztig datum $n_\bullet$, so that $P$ is $\sigma$-invariant. We call such an $n_\bullet$ a $\sigma$-invariant **i**-Lusztig datum.

**Proposition 3.5.** *Let $w_0=s_{\eta_1}s_{\eta_2}\cdots s_{\eta_m}$ be a reduced expression of $w_0$ relative to the Coxeter group $W^\sigma$, where $\eta_1,\eta_2,\cdots,\eta_m$, are orbits of $\sigma$ in $I$. For each $\eta$, we fix a reduced expression of $s_\eta$ as an element of $W$, and denote by* $\mathbf{i}$ *the resulting reduced expression of $w_0$ relative to $W$. Let $n_\bullet$ be the* $\mathbf{i}$*-Lusztig datum of $P$. Then $P$ is $\sigma$-invariant if and only if $n_1=n_2=\cdots=n_{r_{\eta_1}}$, $n_{r_{\eta_1}+1}=n_{r_{\eta_1}+2}=\cdots=n_{r_{\eta_1}+r_{\eta_2}},\cdots$, where $r_\eta$ is the length of $s_\eta$ as an element of $W$.*

*Proof.* For any orbit $\eta$ of $\sigma$, let $R_\eta$ be the root system generated by $\{\alpha_i; i\in\eta\}$. Let $W_\eta$ be the Coxeter group generated by $\{s_i,\text{ for } i\in\eta\}$. Then $s_\eta$ is the longest element in $W_\eta$.

Recall that $n_k$ means the length of the edge connecting $\mu_{w^{\mathbf{i}}_{k-1}}$ with $\mu_{w^{\mathbf{i}}_k}$, i.e., $\mu_{w^{\mathbf{i}}_k}-\mu_{w^{\mathbf{i}}_{k-1}}=-n_k.w^{\mathbf{i}}_{k-1}(\alpha^\vee_{i_k})$. The convex hull of $\{\mu_w|w\in W_{\eta_1}\}$ forms an MV polytope for an algebraic group of type $R_{\eta_1}$. We denote it by $P^1_{\eta_1}$. From $\mu_{w^{\mathbf{i}}_0},\cdots,\mu_{w^{\mathbf{i}}_{r_{\eta_1}}}$, we get a Lusztig datum $(n_1,n_2,\cdots,n_{r_{\eta_1}})$ along the chosen reduced word of $s_\eta$. The convex hull of $\{\mu_w|w=s_{\eta_1}y,\text{ for } y\in W_{\eta_2}\}$ forms an MV polytope of type $R_{\eta_2}$. We denote it by $P^2_{\eta_2}$. From $\mu_{w^{\mathbf{i}}_{r_{\eta_1}+1}},\cdots,\mu_{w^{\mathbf{i}}_{r_{\eta_1}+r_{\eta_2}}}$, we get a Lusztig datum $(n_{r_{\eta_1}+1},n_{r_{\eta_1}+2},\cdots,n_{r_{\eta_1}+r_{\eta_2}})$ along the chosen reduced word of $s_{\eta_2}$. Similarly, we get subsequently MV polytopes $P^3_{\eta_3},\cdots,P^m_{\eta_m}$, with type $R_{\eta_3},\cdots,R_{\eta_m}$. We also get their corresponding Lusztig data along the chosen reduced words of $s_{\eta_i}$.

Now let us return to the proof. If $P$ is $\sigma$-invariant, we have $\sigma(\mu_w)=\mu_{\sigma(w)}$, for all $w\in W$, by Lemma 3.4. Applying Lemma 3.4 again, we see that $P^k_{\eta_k}$, for all $k$, are $\sigma$-invariant.

Note that there are two possibilities: $A_2$ and $A_1\times A_1\times\cdots\times A_1$ ( with $l$ copies of $A_1$, where $\ell=2$ or 3) for $R_\eta$. Hence the sufficient part is reduced to the following two cases which are easy to check.

1. (1) $A_2$, if $P$ is $\sigma$-invariant, then $n_1=n_2=n_3$ .
2. (2) $A_1\times A_1\times\cdots\times A_1$, if $P$ is $\sigma$-invariant, then $n_1=n_2=\cdots=n_l$.

Conversely, from $A^{\mathbf{i}}(n_\bullet)=\bigcap_k S^{\mu_{w^{\mathbf{i}}_k}}_{w^{\mathbf{i}}_k}$, we have $\sigma(A^{\mathbf{i}}(n_\bullet))=A^{\mathbf{j}}(n_\bullet)$, where $\mathbf{j}=(\sigma(i_1),\sigma(i_2),\cdots,\sigma(i_m))$. From the condition of $n_\bullet$, it is easy to see $R^{\mathbf{j}}_{\mathbf{i}}(n_\bullet)=n_\bullet$. Hence their closures coincide, i.e., the corresponding MV cycle of this **i**-Lusztig datum is $\sigma$-invariant. By Lemma 3.4, $P$ is $\sigma$-invariant. ∎

3.4. **The bijection between MV cycles (polytopes) for a symmetric pair.** Let $P$ be a $\sigma$-invariant MV polytope for $G$. In this subsection, we will show that $P^\sigma$ is an MV polytope for $G^\sigma$, and then we get the bijection between MV polytopes for a symmetric pair.

Consider the symmetric pair $(A_4, B_2)$. For the longest element in the Weyl group $W$, we have reduced expressions $w_0 = s_1s_4{\cdot}s_2s_3s_2{\cdot}s_1s_4{\cdot}s_2s_3s_2 = s_2s_3s_2{\cdot}s_1s_4{\cdot}s_2s_3s_2{\cdot}s_1s_4$. We get two reduced words $\mathbf{i}_\sigma$ and $\mathbf{i}'_\sigma$ for $G^\sigma$ from these two expressions of $w_0$. From $\mathbf{i}_\sigma$, and $\mathbf{i}'_\sigma$, we naturally get 2 reduced words for $G$, $\mathbf{i} = (1,4,2,3,2,1,4,2,3,2)$, $\mathbf{i}' = (2,3,2,1,4,2,3,2,1,4)$, respectively. Let $n_\bullet$, $n'_\bullet$ be Lusztig data along $\mathbf{i}$, and $\mathbf{i}'$ for $P$, respectively. According to Proposition 3.5, we may write $n_\bullet$ and $n'_\bullet$ as

$$n_\bullet = (\bar{n}_1, \bar{n}_1, \bar{n}_2, \bar{n}_2, \bar{n}_2, \bar{n}_3, \bar{n}_3, \bar{n}_4, \bar{n}_4, \bar{n}_4) \in \mathbb{N}^{10},$$
$$n'_\bullet = (\bar{n}'_1, \bar{n}'_1, \bar{n}'_1, \bar{n}'_2, \bar{n}'_2, \bar{n}'_3, \bar{n}'_3, \bar{n}'_3, \bar{n}'_4, \bar{n}'_4) \in \mathbb{N}^{10},$$

where $\bar{n}_k$, $\bar{n}'_k$ are nonnegative integers.

Set $n^\sigma_\bullet = (\bar{n}_1, \bar{n}_2, \bar{n}_3, \bar{n}_4)$. By sending $n_\bullet$ to $n^\sigma_\bullet$, we get a bijection between $\mathbf{i}$-Lusztig data of $\sigma$-invariant MV polytopes for $G$ and $\mathbf{i}_\sigma$-Lusztig data of MV polytopes for $G^\sigma$. We shall show this bijection is intrinsic, and independent of the choice of reduced words. Note that the above procedure works for the general case.

For any subvariety $Y \subset \mathcal{G}$, we set $Y^\sigma := \{y \in Y | \sigma(y) = y\}$.

Let $B(n_\bullet) = \{(b_\bullet) \in \mathcal{K}^{\ell(w_0)} | \mathrm{val}(b_k) = n_k,\ \forall k\}$ and $B_\sigma(n^\sigma_\bullet) = \{(b_\bullet) \in \mathcal{K}^{\ell_\sigma(w_0)} | \mathrm{val}(b_k) = \bar{n}_k,\ \forall\ k\}$, where val is the valuation function on $\mathcal{K}$. Define a map $j_\sigma$ from $B_\sigma(n^\sigma_\bullet)$ to $B(n_\bullet)$, by $j_\sigma(b_1, b_2, b_3, b_4) = (b_1, b_1, b_2, 2b_2, b_2, b_3, b_3, b_4, 2b_4, b_4)$.

In this subsection, we always assume that $\mathbf{i}$ and $\mathbf{i}'$ are reduced words of $G$ resulting from the reduced words of $G^\sigma$, $\mathbf{i}_\sigma$ and $\mathbf{i}'_\sigma$, respectively, in the sense of Proposition 3.5.

**Lemma 3.6.** *Let $n_\bullet$ be a $\sigma$-invariant $\mathbf{i}$-Lusztig datum. Then $A^{\mathbf{i}}(n_\bullet)^\sigma = A^{\mathbf{i}_\sigma}(n^\sigma_\bullet)$.*

*Proof.* We only show this lemma for the pair $(A_4, B_2)$, and the following argument works in general.

Let $\iota : A^{\mathbf{i}_\sigma}(n^\sigma_\bullet) \hookrightarrow \mathcal{G}$ be the natural imbedding, which is the restriction of $\iota : \mathcal{G}_\sigma \hookrightarrow \mathcal{G}$. We have surjections $\pi_{\mathbf{i}_\sigma} : B_\sigma(n^\sigma_\bullet) \to A^{\mathbf{i}_\sigma}(n^\sigma_\bullet)$, and $\pi_{\mathbf{i}} : B(n_\bullet) \to A^{\mathbf{i}}(n_\bullet)$, which are given by

$$\pi_{\mathbf{i}_\sigma}(b_1, b_2, b_3, b_4) = [\eta_{w_0}^{-1}(x_{\eta_1}(b_1)x_{\eta_2}(b_2)x_{\eta_1}(b_3)x_{\eta_2}(b_4))],$$
$$\begin{aligned}&\pi_{\mathbf{i}}(b_1, b_1, b_2, 2b_2, b_2, b_3, b_3, b_4, 2b_4, b_4)\\ &= [\eta_{w_0}^{-1}(x_1(b_1)x_4(b_1)\cdot x_2(b_2)x_3(2b_2)x_2(b_2)\cdot x_1(b_3)x_4(b_3)\\ &\quad \cdot x_2(b_4)x_3(2b_4)x_2(b_4))],\end{aligned}$$

where $x_{\eta_1}$ and $x_{\eta_2}$ are root subgroup homomorphisms for $G^\sigma$, and we denote by $[\ ]$ the projection from $G(\mathcal{K})$ to $\mathcal{G}$. For the definition of $\eta_{w_0}$, see section 4.4, [K1]. Since $x_1(b_i)x_4(b_i) = x_{\eta_1}(b_i)$, for i=1 or 3, and $x_2(b_j)x_3(2b_j)x_2(b_j) = x_{\eta_2}(b_j)$, for j=2 or 4, we can see that $\iota \circ \pi_{\mathbf{i}_\sigma} = \pi_{\mathbf{i}} \circ j_\sigma$, i.e., we have the following commutative diagram:

$$\begin{array}{ccc} B_\sigma(n^\sigma_\bullet) & \xrightarrow{\ j_\sigma\ } & B(n_\bullet) \\ \Big\downarrow{\scriptstyle \pi_{\mathbf{i}_\sigma}} & & \Big\downarrow{\scriptstyle \pi_{\mathbf{i}}} \\ A^{\mathbf{i}_\sigma}(n^\sigma_\bullet) & \xrightarrow{\ \iota\ } & A^{\mathbf{i}}(n_\bullet). \end{array}$$

Since $\pi_{\mathbf{i}_\sigma}(B_\sigma(n^\sigma_\bullet)) = A^{\mathbf{i}_\sigma}(n^\sigma_\bullet)$, we have $A^{\mathbf{i}_\sigma}(n^\sigma_\bullet) \subset A^{\mathbf{i}}(n_\bullet)^\sigma$.

Assume $n_\bullet$ is of coweight $\mu$. It is known that $X(\mu) = S_e^0 \cap S_{w_0}^\mu = \bigsqcup A^{\mathbf{i}}(n'_\bullet)$, where the union is taken over $n'_\bullet$, such that $n'_\bullet$ is an $\mathbf{i}'$-Lusztig datum with coweight $\mu$. Hence we have

$$X(\mu)^\sigma = \bigsqcup A^{\mathbf{i}}(n'_\bullet)^\sigma, \tag{1}$$

where $A^{\mathbf{i}}(n_\bullet)$ appear in the right-hand side.

From Corollary 3.3, we have the decomposition

$$X(\mu)^\sigma = \bigsqcup A^{\mathbf{i}_\sigma}(m_\bullet), \tag{2}$$

where the union is taken over $m_\bullet$ such that $m_\bullet$ is an $\mathbf{i}_\sigma$-Lusztig datum with coweight $\mu$.

Let $m_\bullet = (m_1, m_2, m_3, m_4)$ be an $\mathbf{i}_\sigma$-Lusztig datum, such that $\overline{A^{\mathbf{i}_\sigma}(m_\bullet)}$ is an MV cycle for $G^\sigma$ with coweight $\mu$. Let $n''_\bullet = (m_1, m_1, m_2, m_2, m_2, m_3, m_3, m_4, m_4, m_4)$. Then $n''_\bullet$ is $\sigma$-invariant, and hence $A^{\mathbf{i}_\sigma}(m_\bullet) \subset A^{\mathbf{i}}(n''_\bullet)^\sigma$. By comparing decompositions of $X(\mu)^\sigma$ in (1) and (2), we obtain $A^{\mathbf{i}}(n_\bullet)^\sigma = A^{\mathbf{i}_\sigma}(n^\sigma_\bullet)$. □

*Remark* 3.1. From this lemma, we see that the closure of the fixed point set of $\sigma$ on some open subset of a $\sigma$-invariant MV cycle $C$ is an MV cycle for $G^\sigma$. We believe that the fixed point set of $\sigma$ on a $\sigma$-invariant MV cycle for $G$ is an MV cycle for $G^\sigma$.

**Corollary 3.7.** *If $\overline{A^{\mathbf{i}}(n_\bullet)}$ is not $\sigma$-invariant, then $A^{\mathbf{i}}(n_\bullet)^\sigma$ is empty.*

**Lemma 3.8.** *If $n_\bullet$ is a $\sigma$-invariant $\mathbf{i}$-Lusztig datum, and $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet) = n'_\bullet$, then $(A^{\mathbf{i}}(n_\bullet) \cap A^{\mathbf{i}'}(n'_\bullet))^\sigma$ contains an open dense subset.*

*Proof.* We can change $\mathbf{i}$ to $\mathbf{i}'$ by combining several braid $d$-moves.

If $(\cdots, i_k, i_{k+1}, i_{k+2}, i_{k+3}, \cdots) \mapsto (\cdots, i_k, i_{k+2}, i_{k+1}, i_{k+3}, \cdots)$, $(d = 2)$, define a rational map from $B(n_\bullet)$ to $B(n'_\bullet)$, by

$$(\cdots, b_k, b_{k+1}, b_{k+2}, b_{k+3}, \cdots) \mapsto (\cdots, b_k, b_{k+2}, b_{k+1}, b_{k+3}, \cdots).$$

If $(\cdots, i_k, i_{k+1}, i_{k+2}, i_{k+3}, i_{k+4}, \cdots) \mapsto (\cdots, i_k, i_{k+2}, i_{k+1}, i_{k+2}, i_{k+4}, \cdots)$, $(d = 3)$, where $i_{k+1} = i_{k+3}$, then we define a rational map from $B(n_\bullet)$ to $B(n'_\bullet)$ by

$$(\cdots, b_k, b_{k+1}, b_{k+2}, b_{k+3}, b_{k+4} \cdots) \\ \mapsto (\cdots, b_k, \frac{b_{k+2}b_{k+3}}{b_{k+1}+b_{k+3}}, b_{k+1}+b_{k+3}, \frac{b_{k+1}b_{k+2}}{b_{k+1}+b_{k+3}}, b_{k+4}, \cdots).$$

It is well known that, by several braid $d$-moves, we can arrive at $\mathbf{i}'$ from $\mathbf{i}$. Let $\mathbf{i} \mapsto \mathbf{i}_1 \mapsto \mathbf{i}_2 \mapsto \cdots \mapsto \mathbf{i}'$ be one such path, where $\mapsto$ represents a braid $d$-move. For a path from $\mathbf{i}$ to $\mathbf{i}'$, we denote the rational map $f$ by combining those in every step defined above. Assume $f(b_1, \cdots, b_m) = (b'_1, \cdots, b'_m)$. It is easy to see that $b'_k$ is a rational function with numerator and denominator as nonzero polynomials with nonnegative integral coefficients. Consider the diagram

$$\begin{array}{ccc} B(n_\bullet) & \dashrightarrow & B(n'_\bullet) \\ \downarrow \pi_{\mathbf{i}} & & \downarrow \pi_{\mathbf{i}'} \\ A^{\mathbf{i}}(n_\bullet) & \dashrightarrow & A^{\mathbf{i}'}(n'_\bullet) \end{array}$$

where $\pi_{\mathbf{i}}$ is as in the proof of Lemma 3.6, and dashed arrows denote rational maps. We have $\pi_{\mathbf{i}} = \pi_{\mathbf{i}'} \circ f$.

Let $F$ be the product of all denominators appearing in every step of $d$-moves, so it is a nonzero polynomial with nonnegative integral coefficients. Let $U = \{(b_\bullet) \in B(n_\bullet) | F(b_\bullet) \neq 0\}$. Then $f$ is well defined on $U$, and so $\pi_{\mathbf{i}}(U) \subset A^{\mathbf{i}}(n_\bullet) \cap A^{\mathbf{i}'}(n'_\bullet)$.

There exists $y \in U$, such that $\pi_{\mathbf{i}}(y) \in \pi_{\mathbf{i}}(U) \subset A^{\mathbf{i}}(n_\bullet) \cap A^{\mathbf{i}'}(n'_\bullet)$, and $\pi_{\mathbf{i}}(y)$ is $\sigma$-invariant. Hence $(A^{\mathbf{i}}(n_\bullet) \cap A^{\mathbf{i}'}(n'_\bullet))^\sigma$ is nonempty. Since $\pi_{\mathbf{i}}$ is an open map, $\pi_{\mathbf{i}}(U)$ is open in $A^{\mathbf{i}}(n_\bullet)$. We only show it in the case of $(A_4, B_2)$. Since $\overline{A^{\mathbf{i}}(n_\bullet)}$ is $\sigma$-invariant, we have $n_\bullet = (\bar{n}_1, \bar{n}_1, \bar{n}_2, \bar{n}_2, \bar{n}_2, \bar{n}_3, \bar{n}_3, \bar{n}_4, \bar{n}_4, \bar{n}_4)$. Now take $y = (t^{\bar{n}_1}, t^{\bar{n}_1}, t^{\bar{n}_2}, 2t^{\bar{n}_2}, t^{\bar{n}_2}, t^{\bar{n}_3}, t^{\bar{n}_3}, t^{\bar{n}_4}, 2t^{\bar{n}_4}, t^{\bar{n}_4}) \in B(n_\bullet)$, then $F(y) \neq 0$. In the general case, we have a similar argument.

Since $A^{\mathbf{i}}(n_\bullet)$ is irreducible by Lemma 3.6, we have $(A^{\mathbf{i}}(n_\bullet) \cap A^{\mathbf{i}'}(n'_\bullet))^\sigma$ is dense in $A^{\mathbf{i}}(n_\bullet)^\sigma$. □

**Lemma 3.9.** *Let* $\mathrm{Conv}((\mu_w)_{w\in W^\sigma})$ *be the convex hull of* $(\mu_w)_{w\in W^\sigma}$ *in* $t_{\mathbb{R}}$. *If the MV polytope* $P = \mathrm{Conv}((\mu_w)_{w\in W})$ *is* $\sigma$*-invariant, then* $P^\sigma = \mathrm{Conv}((\mu_w)_{w\in W^\sigma})$.

*Proof.* Since $P$ is $\sigma$-invariant, we have $\sigma(\mu_w) = \mu_w$, for $w \in W^\sigma$. We can easily see that $\sigma$ acts trivially on $\mathrm{Conv}((\mu_w)_{w\in W^\sigma})$, so $\mathrm{Conv}((\mu_w)_{w\in W^\sigma}) \subset P^\sigma$.

For the converse, the perfect pairing $(X_* \otimes \mathbb{R}) \times (X^* \otimes \mathbb{R}) \to \mathbb{R}$ descends to $(X_*^\sigma \otimes \mathbb{R}) \times (X_\sigma^* \otimes \mathbb{R}) \to \mathbb{R}$ (see Section 2.2). Note that $t_{\mathbb{R}}^\sigma$ can be identified with $X_*^\sigma \otimes \mathbb{R}$.

For any $\beta \in P^\sigma \subset P$, and $w \in W^\sigma$, we have $\langle \beta, w\cdot\lambda_i\rangle \leq \langle \mu_w, w\cdot\lambda_i\rangle$. By descent, we have $\langle \beta, w\cdot\lambda_\eta\rangle \leq \langle \mu_w, w\cdot\lambda_\eta\rangle$, for every orbit $\eta$ of $\sigma$ in $I$, where $\lambda_\eta$ is the fundamental weight for $G^\sigma$ corresponding to $\lambda_i$, for $i \in I$. Since $P^\sigma \subset t_{\mathbb{R}}^\sigma$, we see that

$$P^\sigma \subset \{\beta \in t_{\mathbb{R}}^\sigma | \langle \beta, w\cdot\lambda_\eta\rangle \leq \langle \mu_w, w\cdot\lambda_\eta\rangle,\ \forall \eta,\ \forall\ w \in W^\sigma\}.$$

The right-hand side is exactly $\mathrm{Conv}((\mu_w)_{w\in W^\sigma})$. □

**Theorem 3.10.** *If* $P$ *is a* $\sigma$*-invariant MV polytope for* $G$*, then* $P^\sigma$ *is an MV polytope for* $G^\sigma$.

*Proof.* Let $\mu_\bullet$ be the vertices of $P$. Fix a reduced word $\mathbf{i}_\sigma$ for $G^\sigma$, and let $n_\bullet^\sigma$ be the corresponding $\mathbf{i}_\sigma$-Lusztig datum of $P$.

Let $\mathbf{i}$ be the fixed reduced word for $G$ from $\mathbf{i}_\sigma$, in the sense of Proposition 3.5. Let $J = \{(\mathbf{i}', n'_\bullet) | \mathbf{i}'$ be a reduced word for $G$ from some reduced word $\mathbf{i}'_\sigma$ for $G^\sigma$, and $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet) = n'_\bullet\}$. We have $\bigcap_{(\mathbf{i}', n'_\bullet)\in J} A^{\mathbf{i}'}(n'_\bullet)^\sigma$ contains an open and dense subset of $A^{\mathbf{i}}(n_\bullet)^\sigma$ from Lemma 3.8, since the intersection of finite open dense subsets is still open and dense.

Recall $A^{\mathbf{i}'}(n'_\bullet) = \bigcap S_{w_k^{\mathbf{i}'}}^{\mu_{w_k^{\mathbf{i}'}}}$, and $A^{\mathbf{i}'_\sigma}(n'^\sigma_\bullet) = \bigcap (S_\sigma)_{w_k^{\mathbf{i}'_\sigma}}^{\mu_{w_k^{\mathbf{i}'_\sigma}}}$. By Lemma 3.6, we have $(\bigcap_{(\mathbf{i}', n'_\bullet)\in J} A^{\mathbf{i}'}(n'_\bullet))^\sigma = \bigcap_{(\mathbf{i}'_\sigma, n'^\sigma_\bullet)} A^{\mathbf{i}'_\sigma}(n'^\sigma_\bullet) = A((\mu_w)_{w\in W^\sigma})$, where $A((\mu_w)_{w\in W^\sigma}) = \bigcap_{w\in W^\sigma} (S_\sigma)_w^{\mu_w}$. The last equality holds, since for any $w \in W^\sigma$, there exists some reduced word $\mathbf{i}'_\sigma$ of $G^\sigma$ and some integer $k$, such that $w = w_k^{\mathbf{i}'_\sigma}$. Therefore, we have $\overline{A^{\mathbf{i}_\sigma}(n_\bullet^\sigma)} = \overline{A^{\mathbf{i}}(n_\bullet)^\sigma} = \overline{(\bigcap_{(\mathbf{i}', n'_\bullet)\in J} A^{\mathbf{i}'}(n'_\bullet))^\sigma} = \overline{A((\mu_w)_{w\in W^\sigma})}$. That means, the moment polytope of the MV cycle $\overline{A^{\mathbf{i}_\sigma}(n_\bullet^\sigma)}$ is $\mathrm{Conv}((\mu_w)_{w\in W^\sigma})$, which is exactly $P^\sigma$, by Lemma 3.9. Hence $P^\sigma$ is really an MV polytope for $G^\sigma$. □

**Corollary 3.11.** *Let* $(\mathbf{i}, n_\bullet)$ *and* $(\mathbf{i}', n'_\bullet)$ *be two* $\sigma$*-invariant Lusztig data. If* $R_{\mathbf{i}}^{\mathbf{i}'}(n_\bullet) = n'_\bullet$, *then* $R_{\mathbf{i}_\sigma}^{\mathbf{i}'_\sigma}(n_\bullet^\sigma) = n'^\sigma_\bullet$

**Theorem 3.12.** *We have a bijection* $\theta_P : \mathrm{MVP}_G^\sigma \longrightarrow \mathrm{MVP}_{G^\sigma}$*, given by* $P \mapsto P^\sigma$*, which preserves coweights. Induced from* $\theta_P$*, we have a bijection* $\theta_C : \mathrm{MVC}_G^\sigma \longrightarrow \mathrm{MVC}_{G^\sigma}$

*Proof.* Let $P$ be a $\sigma$-invariant MV polytope for $G$. By Theorem 3.10, we have a well-defined map $\theta_P : \mathrm{MVP}_G^\sigma \longrightarrow \mathrm{MVP}_{G^\sigma}$ by $\theta_P(P) = P^\sigma$.

Fix a reduced word $\mathbf{i}_\sigma$ for $G^\sigma$. Let $\mathbf{i}$ be a reduced word coming from $\mathbf{i}_\sigma$. For any MV polytope for $G$ (resp. $G^\sigma$), we have the corresponding $\mathbf{i}$ (resp. $\mathbf{i}_\sigma$) Lusztig datum. According to Proposition 3.5, $\theta_P$ is injective. Let $Q$ be any MV polytope for $G^\sigma$, and let $m_\bullet$ be the $\mathbf{i}_\sigma$-Lusztig datum of $Q$. By Lemma 3.6 and its proof, there exists a unique $\mathbf{i}$-Lusztig datum $n_\bullet$ such that $A^{\mathbf{i}_\sigma}(m_\bullet)$ is contained in $A^{\mathbf{i}}(n_\bullet)$, and $n_\bullet$ is $\sigma$-invariant. Let $P_Q$ be the MV polytope of $\overline{A^{\mathbf{i}}(n_\bullet)}$. We have $P_Q^\sigma = Q$, since $P_Q^\sigma$ has the same $\mathbf{i}_\sigma$-Lusztig datum as $Q$. So $\theta_P$ is surjective.

Hence $\theta_P$ is a bijection, and it is easy to see that it preserves the coweights of MV polytopes. □

3.5. **The bijection in the highest weight case.** Let $\lambda$, $\mu$ be $\sigma$-invariant coweights, we set $X(\lambda,\mu) := S_e^\lambda \cap S_{w_0}^\mu$, and $X(\mu-\lambda) = S_e^0 \cap S_{w_0}^{\mu-\lambda}$. In this subsection, we have the same assumptions on $\mathbf{i}$ and $\mathbf{i}_\sigma$ as in Subsection 3.4.

The following lemma is given by Anderson [A]

**Lemma 3.13.** *An irreducible component of* $X(\lambda,\mu)$ *is contained in* $\overline{\mathcal{G}^\lambda}$ *if and only if it appears as basis in* $V_\mu(\lambda)$

First, we have the decomposition

$$X(\lambda,\mu) = \lambda\cdot X(\mu-\lambda) = \bigsqcup \lambda\cdot A^{\mathbf{i}}(n_\bullet), \tag{3}$$

where the union is taken over $n_\bullet$ which are $\mathbf{i}$-Lusztig data with coweight $\mu-\lambda$. Then

$$S_e^\lambda \cap S_{w_0}^\mu \cap \overline{\mathcal{G}^\lambda} = \bigsqcup_1 \lambda\cdot A^{\mathbf{i}}(n_\bullet) \cup \bigsqcup_2 (\lambda\cdot A^{\mathbf{i}}(n_\bullet)\cap \overline{\mathcal{G}^\lambda}), \tag{4}$$

where the first union 1 is taken over those $n_\bullet$ in (3) such that $\lambda\cdot A^{\mathbf{i}}(n_\bullet)\subset\overline{\mathcal{G}^\lambda}$; the second union 2 is taken over those $n_\bullet$ in (3) such that $\lambda\cdot A^{\mathbf{i}}(n_\bullet)\nsubseteq\overline{\mathcal{G}^\lambda}$.

If $\lambda\cdot A^{\mathbf{i}}(n_\bullet)\nsubseteq\overline{\mathcal{G}^\lambda}$, then $\lambda\cdot A^{\mathbf{i}}(n_\bullet)\cap\overline{\mathcal{G}^\lambda}$ is of lower dimension than $A^{\mathbf{i}}(n_\bullet)$.

From decomposition (4) and Corollary 3.7, we have

$$(S_e^\lambda\cap S_{w_0}^\mu\cap\overline{\mathcal{G}^\lambda})^\sigma = (S_e^\lambda)^\sigma\cap(S_{w_0}^\mu)^\sigma\cap(\overline{\mathcal{G}^\lambda})^\sigma = \bigsqcup_3 \lambda\cdot A^{\mathbf{i}}(n_\bullet)^\sigma \cup \bigsqcup_4 (\lambda\cdot A^{\mathbf{i}}(n_\bullet)\cap\overline{\mathcal{G}^\lambda})^\sigma, \tag{5}$$

where the first union 3 is taken over those $n_\bullet$ in (3), such that $\lambda\cdot A^{\mathbf{i}}(n_\bullet)\subset\overline{\mathcal{G}^\lambda}$ and $n_\bullet$ is $\sigma$-invariant; the second union 4 is taken over those $n_\bullet$ in (3), such that $\lambda\cdot A^{\mathbf{i}}(n_\bullet)\nsubseteq\overline{\mathcal{G}^\lambda}$ and $n_\bullet$ is $\sigma$-invariant. From the viewpoint of $G^\sigma$, we also have the decomposition

$$(S_\sigma)_e^\lambda\cap(S_\sigma)_{w_0}^\mu\cap(\overline{\mathcal{G}^\lambda})^\sigma = \bigsqcup_5 \lambda\cdot A^{\mathbf{i}_\sigma}(m_\bullet)\cup\bigsqcup_6(\lambda\cdot A^{\mathbf{i}_\sigma}(m_\bullet)\cap\overline{\mathcal{G}_\sigma^\lambda}), \tag{6}$$

where the first union 5 is taken over $m_\bullet$ which are $\mathbf{i}_\sigma$-Lusztig data with coweight $\mu-\lambda$, satisfying $\lambda\cdot A^{\mathbf{i}_\sigma}(m_\bullet)\subset\overline{\mathcal{G}_\sigma^\lambda}$; the second union 6 is taken over $m_\bullet$ which are $\mathbf{i}_\sigma$-Lusztig data with coweight $\mu-\lambda$, satisfying $\lambda\cdot A^{\mathbf{i}_\sigma}(m_\bullet)\nsubseteq\overline{\mathcal{G}_\sigma^\lambda}$.

If $\lambda\cdot A^{\mathbf{i}_\sigma}(m_\bullet)\nsubseteq\overline{\mathcal{G}^\lambda}$, then $\lambda\cdot A^{\mathbf{i}_\sigma}(m_\bullet)\cap\overline{\mathcal{G}_\sigma^\lambda}$ is of lower dimension than $A^{\mathbf{i}_\sigma}(m_\bullet)$.

**Lemma 3.14.** $\overline{\mathcal{G}^\lambda} = \overline{\bigcap S_w^{w\cdot\lambda}}$.

*Proof.* We know that $\overline{\bigcap S_w^{w\cdot\lambda}}$ is an MV cycle with coweight $(\lambda, w_0\cdot\lambda)$, and it is contained in $\overline{\mathcal{G}^\lambda}$. Since both of them are of the same dimension $2\langle\lambda,\rho\rangle$, and both of them are irreducible, we have $\overline{\mathcal{G}^\lambda} = \overline{\bigcap S_w^{w\cdot\lambda}}$. □

**Lemma 3.15.** *If $\lambda\cdot A^{\mathbf{i}}(n_\bullet) \nsubseteq \overline{\mathcal{G}^\lambda}$, and $n_\bullet$ is $\sigma$-invariant, then $(\lambda\cdot A^{\mathbf{i}}(n_\bullet)\cap\overline{\mathcal{G}^\lambda})^\sigma$ is of lower dimension than $A^{\mathbf{i}}(n_\bullet)^\sigma$.*

*Proof.* With the same reason as in the proof of Lemma 3.6, we can find an open subset $U\subset B(n_\bullet)$, such that $\pi_{\mathbf{i}}(U)\subset\bigcap_{(\mathbf{i},n_\bullet)} A^{\mathbf{i}}(n_\bullet) = \bigcap_w S_w^{\mu_w}$ is open in $A^{\mathbf{i}}(n_\bullet)$.

Note that $(\bigcap\lambda\cdot S_w^{\mu_w})\cap\overline{\mathcal{G}^\lambda}$ is empty. Otherwise, if there exists a point $p\in(\cap\lambda\cdot S_w^{\mu_w})\cap\overline{\mathcal{G}^\lambda}$, then

$$p\in(\bigcap\lambda\cdot S_w^{\mu_w})\cap\overline{\mathcal{G}^\lambda} = (\bigcap\lambda\cdot S_w^{\mu_w})\cap\overline{\cap S_w^{w\cdot\lambda}}\subset(\bigcap\lambda\cdot S_w^{\mu_w})\cap\overline{S_w^{w\cdot\lambda}}.$$

That is, $\forall\ w\in W$, $p$ must be contained in $\lambda\cdot S_w^{\mu_w}\cap\overline{S_w^{w\cdot\lambda}}$. From $\overline{S_w^{w\cdot\lambda}} = \bigsqcup_{\mu\leq_w w\cdot\lambda} S_w^\mu$, we have $\mu_w+\lambda\leq_w w\cdot\lambda$. We get that $\mathrm{Conv}(\mu_\bullet)+\lambda\subset\mathrm{Conv}(W\cdot\lambda)$. According to Anderson's theorem on multiplicity of weight space [A], we have $\lambda\cdot\overline{A(\mu_\bullet)}$ is an MV cycle in $V_\mu(\lambda)$. By Lemma 3.13, it is a contradiction to the condition that $\lambda\cdot A^{\mathbf{i}}(n_\bullet)\nsubseteq\overline{\mathcal{G}^\lambda}$. As in Lemma 3.8, there exists a point $p\in\lambda\cdot A^{\mathbf{i}}(n_\bullet)$. So $\lambda\cdot A^{\mathbf{i}}(n_\bullet)^\sigma\cap\overline{\mathcal{G}^\lambda}^\sigma$ has lower dimension than $A^{\mathbf{i}}(n_\bullet)^\sigma$. □

By Lemma 3.15, and by comparing the two decompositions (5) and (6), we have that the set $\{A^{\mathbf{i}}(n_\bullet)|n_\bullet$ is $\sigma$-invariant and is of coweight $\mu-\lambda$, and $\lambda\cdot A^{\mathbf{i}}(n_\bullet)\subseteq\overline{\mathcal{G}^\lambda}\}$ is in bijection with the set $\{A^{\mathbf{i}_\sigma}(m_\bullet)|m_\bullet$ is of coweight $\mu-\lambda$, and $\lambda\cdot A^{\mathbf{i}_\sigma}(m_\bullet)\subseteq\overline{\mathcal{G}_\sigma^\lambda}\}$, by sending $A^{\mathbf{i}}(n_\bullet)$ to $A^{\mathbf{i}}(n_\bullet)^\sigma$. We thus obtain the following theorem.

**Theorem 3.16.** *We have a bijection $\theta_C^\lambda:\mathrm{MVC}_G(\lambda)^\sigma\longrightarrow\mathrm{MVC}_{G_\sigma}(\lambda)$, which is the restriction of $\theta_C$ in Theorem 3.12.*

## 4. Twining character formula

Recall that $\mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G})$ is a tensor category [MV], and it is easy to see the tensor functor $\sigma^*$ induced from the action of $\sigma$ on affine Grassmannian is a tensor equivalence. From the functoriality of Tannakian formalism [DM], we have a natural automorphism $\bar\sigma$ on $G^\vee$.

Fix a $\sigma$-invariant coweight $\lambda$, and choose an isomorphism $\phi: IC_\lambda\simeq\sigma^*(IC_\lambda)$, which is compatible with the action of $\sigma$ on MV cycles (as the basis of $V(\lambda)$).

**Lemma 4.1.** *The action of $\bar\sigma$ on $G^\vee$ is compatible with the natural action of $\sigma$ on $V(\lambda)$ induced from $\phi$.*

*Proof.* Let $T$ be the functor from $\mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G})$ to $\mathrm{Rep}(G^\vee)$, such that $T(IC_\lambda) = (\rho_\lambda, V(\lambda))$, where $\rho_\lambda: G^\vee\to GL(V(\lambda))$ is the corresponding representation.

From $\sigma^*:\mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G})\to\mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G})$, we get $T(\sigma^*(IC_\lambda)) = (\rho_\lambda\circ\bar\sigma, V(\lambda))$. Let $\tilde\sigma$ be the functor from $\mathrm{Rep}(G^\vee)$ to $\mathrm{Rep}(G^\vee)$, by sending $(\rho_\lambda, V(\lambda))$ to $(\rho_\lambda\circ\bar\sigma, V(\lambda))$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G}) & \xrightarrow{T} & \mathrm{Rep}(G^\vee) \\ \Big\downarrow{\scriptstyle\sigma^*} & & \Big\downarrow{\scriptstyle\tilde\sigma} \\ \mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G}) & \xrightarrow{T} & \mathrm{Rep}(G^\vee)\,. \end{array}$$

By applying $T$ to $\phi : IC_\lambda \simeq \sigma^*(IC_\lambda)$, we obtain an isomorphism $\sigma = T(\phi) : (\rho_\lambda, V(\lambda)) \to (\rho_\lambda \circ \bar{\sigma}, V(\lambda))$ in $\mathrm{Rep}(G^\vee)$. In other words, there exists a linear isomorphism $\sigma : V(\lambda) \to V(\lambda)$ satisfying

$$\sigma(\rho_\lambda(g)\cdot v) = (\rho_\lambda \circ \bar{\sigma})(g)\cdot \sigma(v) = \rho_\lambda(\bar{\sigma}(g))\cdot \sigma(v), (g \in G^\vee, v \in V(\lambda)). \qquad \square$$

**Theorem 4.2.** *$\bar{\sigma}$ is a Dynkin automorphism on $G^\vee$.*

*Proof.* Let $\mathrm{Vect}_{X_*}$ be the tensor category of $X_*$-graded vector spaces. The action of $\sigma$ on $X_*$ induces an tensor functor $\sigma^\circ$ on $\mathrm{Vect}_{X_*}$. From Mirkovic-Vilonen's paper [MV], we know that there is a tensor functor $F$ from $\mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G})$ to $\mathrm{Vect}_{X_*}$, and it is easy to see $\sigma^*$ and $\sigma^\circ$ are compatible with $F$.

Applying Tannkian formalism, from $F$ we get the forgetful functor from $\mathrm{Rep}(G^\vee)$ to $\mathrm{Rep}(T^\vee)$, where $T^\vee$ is a torus of $G^\vee$, and $\sigma^*$, $\sigma^\circ$ induce automorphisms on $G^\vee$ and $T^\vee$, respectively. Since $\sigma^*$ and $\sigma^\circ$ are compatible with $F$, we have $\bar{\sigma}$ preserves the torus $T^\vee$, i.e., $\bar{\sigma}(T^\vee) = T^\vee$. It induces the action of $\sigma$ on $X^*(T^\vee)$.

Let $B^\vee$ be the maximal subgroup of $G^\vee$, which stabilizes the highest weight line $V_\lambda(\lambda)$ in $V(\lambda)$, for any $\sigma$-invariant dominant weight $\lambda$ of $G^\vee$. It is easy to see $B^\vee$ is a Borel subgroup of $G$, and contains $T^\vee$. For any $\sigma$-invariant dominant weight $\lambda$, $\sigma$ acts on $V(\lambda)$ by interchanging MV cycles, especially $\sigma$ acts trivially on $V_\lambda(\lambda)$. From Lemma 4.1 and the triviality of $\sigma$ on $V_\lambda(\lambda)$, we have $\bar{\sigma}(b)\cdot V_\lambda(\lambda) = \sigma(b \cdot V_\lambda(\lambda)) = \sigma(V_\lambda(\lambda)) = V_\lambda(\lambda)$, for any $b \in B^\vee$. Hence we have $\bar{\sigma}(B^\vee) = B^\vee$.

The coroots of $G$ $\alpha_i^\vee$, $i \in I$, can be viewed as the roots of $G^\vee$, and $\sigma$ sends the root $\alpha_i^\vee$ to $\alpha_{\sigma(i)}^\vee$ automatically, since under the identification of $X^*(T^\vee)$ and $X_*$, the actions of $\sigma$ are compatible.

Since $\sigma(T^\vee) = T^\vee$ and $\sigma(B^\vee) = B^\vee$, we can see that $\sigma$ maps the root subgroup $U_{\alpha^\vee}$ to $U_{\sigma(\alpha^\vee)}$, where $\alpha^\vee$ is a root of $G^\vee$. In particular, $\sigma(U_{\alpha_i^\vee}) = U_{\alpha_{\sigma(i)}^\vee}$, for any $i \in I$.

Let $\mathscr{G}^\vee$ be the Lie algebra of $G^\vee$. Let $\tau$ be the automorphism on $\mathscr{G}^\vee$ induced from $\bar{\sigma}$. From the following Lemma 4.3, we know $\tau$ acts trivially on the simple root space $\mathscr{G}^\vee_{\alpha_i^\vee}$ and $\mathscr{G}^\vee_{-\alpha_i^\vee}$, for $i$ fixed by $\sigma$. Lift $\tau$ to $\bar{\sigma}$ on $G^\vee$, then $\bar{\sigma}$ acts trivially on the root subgroup $U_{\alpha_i^\vee}$ and $U_{-\alpha_i^\vee}$, for $i$, $\sigma(i) = i$. Hence we are able to find root subgroup homomorphisms $x_i^\vee : \mathbb{C} \to G^\vee$ and $y_i^\vee : \mathbb{C} \to G^\vee$, corresponding to $\alpha_i^\vee$ and $-\alpha_i^\vee$, such that $\bar{\sigma}(x_i^\vee(a)) = x_{\sigma(i)}^\vee(a)$ and $\bar{\sigma}(y_i^\vee(a)) = y_{\sigma(i)}^\vee(a)$, for any $a \in \mathbb{C}$, and for any $i \in I$.

Hence $\bar{\sigma}$ is a Dynkin automorphism with respect to a pinning $(G^\vee, T^\vee, B^\vee, x_i^\vee, y_i^\vee, i \in I)$ of $G^\vee$. $\square$

Assume the highest root is $\gamma^\vee$, then it is $\sigma$-invariant. $\mathscr{G}^\vee$ admits a highest representation of $G^\vee$ with highest weight $\gamma^\vee$. Assume $e_{\alpha^\vee}$ is the basis corresponding to the unique MV cycle in the root space $\mathscr{G}^\vee_{\alpha^\vee}$, for each root $\alpha^\vee$ of $G^\vee$. By interchanging MV cycles, we get a linear operator $\sigma$ on $\mathscr{G}^\vee$, especially $\sigma(e_{\alpha^\vee}) = e_{\sigma(\alpha^\vee)}$. Recall $\tau$ is an automorphism on $\mathscr{G}^\vee$, we have

**Lemma 4.3.** *As linear operators on $\mathscr{G}^\vee$, if $G^\vee$ is of type $A_{2n}$, then $\tau = -\sigma$; otherwise $\tau = \sigma$.*

*Proof.* Let $\mathscr{H}^\vee$ be the Lie algebra of $T^\vee$. It is a Cartan subalgebra of $\mathscr{G}^\vee$, and it can be identified with $X^* \otimes \mathbb{C}$, where the actions of $\tau$ on $\mathscr{H}^\vee$ and $\sigma$ on $X^*$ are compatible.

From Lemma 4.1, we have $\sigma([a,b]) = [\tau(a), \sigma(b)]$, for two arbitrary elements $a$ and $b$ in $\mathscr{G}^\vee$. By Schur's lemma, we have $\tau = c \cdot \sigma$, for some nonzero constant $c$.

Let $\gamma$ be the corresponding coroot of highest root $\gamma^\vee$, so it is $\sigma$-invariant. Since $[e_{\gamma^\vee}, e_{-\gamma^\vee}] \in \mathbb{C} \cdot \gamma$, we have $[e_{\gamma^\vee}, e_{-\gamma^\vee}] = \tau([e_{\gamma^\vee}, e_{-\gamma^\vee}]) = [\tau(e_{\gamma^\vee}), \tau(e_{-\gamma^\vee})] = c^2 \cdot [e_{\gamma^\vee}, e_{-\gamma^\vee}]$. Hence $c^2 = 1$.

If $G^\vee$ is of type $A_{2n}$, there exists two adjacent simple roots $\alpha_i^\vee$ and $\alpha_j^\vee$, such that $\sigma(i) = j$, for $i$ and $j \in I$. Then we have $\tau([e_{\alpha_i^\vee}, e_{\alpha_j^\vee}]) = [e_{\alpha_j^\vee}, e_{\alpha_i^\vee}] = -[e_{\alpha_i^\vee}, e_{\alpha_j^\vee}]$. Since $\alpha_i^\vee + \alpha_j^\vee$ is also $\sigma$-invariant, it forces $c = -1$.

If $G^\vee$ is of another type, then let $h_i = [e_{\alpha_i^\vee}, e_{-\alpha_i^\vee}]$. Then $\{h_i\}_{i \in I}$ is a basis of $\mathscr{H}^\vee$. Since $\sigma([e_{\alpha_i^\vee},\, e_{-\alpha_i^\vee}]) = [\tau(e_{\alpha_i^\vee}),\, \sigma(e_{-\alpha_i^\vee})] = c \cdot [e_{\alpha^\vee_{\sigma(i)}},\, e_{\alpha^\vee_{-\sigma(i)}}]$, we have $\sigma(h_i) = c \cdot h_{\sigma(i)}$. It is easy to see that $\mathrm{trace}(\sigma|_{\mathscr{H}^\vee}) = c \cdot \sharp\{i \in I | \sigma(i) = i\}$. Since there exists $i \in I$, such that $\sigma(i) = i$, when $G^\vee$ is not of type $A_{2n}$, we have $\mathrm{trace}(\sigma|_{\mathscr{H}^\vee}) \neq 0$. Moreover, $\sigma$ interchanges MV cycles in $\mathscr{H}^\vee$, so $\mathrm{trace}(\tau|_{\mathscr{H}^\vee}) \geq 0$. We thus have $c = 1$. □

*Remark* 4.1. We can give another construction of the Dynkin automorphism on $G^\vee$ which is compatible with the action of $\sigma$ on MV cycles, by using Vasserot's explicit construction of the action of the dual group on cohomology of perverse sheaves [V]. Moreover, this automorphism coincides with the one from Tannakian formalism.

We have shown that $\bar{\sigma}$ is a Dynkin automorphism, and from Lemma 4.1, we see that the twining character $\mathrm{ch}^\sigma(V(\lambda)) = \sum_{\mu \in P(\lambda)^\sigma} \mathrm{trace}(\sigma|_{V_\mu(\lambda)})e^\mu$, where $\lambda$ is $\sigma$-invariant.

**Proposition 4.4.**

$$\mathrm{ch}^\sigma(V(\lambda)) = \frac{\sum_{w \in W^\sigma} (-1)^{\ell_\sigma(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W^\sigma} (-1)^{\ell_\sigma(w)} e^{w(\rho)}}.$$

*Proof.* Let $V^\sigma(\lambda)$ be the irreducible representation of $(G^\sigma)^\vee$ with highest weight $\lambda$. By the Weyl character formula for $G^\sigma$, we have

$$\sum_{\mu \in P(\lambda)^\sigma} \dim V_\mu^\sigma(\lambda) e^\mu = \frac{\sum_{w \in W^\sigma} (-1)^{\ell_\sigma(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W^\sigma} (-1)^{\ell_\sigma(w)} e^{w(\rho)}}.$$

Comparing with our definition of twining character for $G^\vee$, we see that it is equivalent to showing that $\mathrm{trace}(\sigma|_{V_\mu(\lambda)}) = \dim V_\mu^\sigma(\lambda)$, for any $\mu \in P(\lambda)^\sigma$. By Lemma 4.1, $\mathrm{trace}(\sigma|_{V_\mu(\lambda)} = \sharp(\mathrm{MVC}_G^\mu(\lambda)^\sigma)$. Hence our proposition follows from Theorem 3.16 □

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