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Fusion rings revisited

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Abstract. In this note we describe a general elementary procedure to attach a fusion ring to any Kac-Moody algebra of affine type. In the case of untwisted affine algebras, they are usual fusion rings in the literature. In the case of twisted affine algebras, they are exactly the twisted fusion rings defined previously by the author via tracing out diagram automorphisms on conformal blocks for appropriate simply-laced Lie algebras. We also relate the fusion ring to the modular S-matrix for any Kac-Moody algebra of affine type.

1. Kac-Moody algebras of affine types

Let A be a generalized Cartan matrix of affine type of order $n+1$ (and rank n). We denote by $a_i(i = 0, 1, \cdots, n)$ the labelling on the vertices of Dynkin diagram associated to A ([**Ka**, §4.8, Table Aff 1, Table Aff 2]). We denote by $\check{a}_i(i = 0, 1, \cdots, n)$ the labelling of the vertices of the affine Dynkin diagram associated to the transpose A^t of A, which is obtained rom the Dynkin diagram of A by reversing all arrows and keeping the same enumeration of vertices.

Let $\mathfrak{g}(A)$ denote the Kac-Moody algebra associated to A with Cartan subalgebra $\mathfrak{h}$, roots $\alpha_i(i = 0, 1, \cdots n)$ and coroots $\check{\alpha}_i(i = 0, 1, \cdots, n)$. Note that $A = (\langle \alpha_i, \check{\alpha}_j \rangle)$,where $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathfrak{h}$ and its dual $\mathfrak{h}^*$. Let $e_i, f_i\,(i = 0, 1, \cdots, n)$ denote the Chevalley generators of $\mathfrak{g}(A)$. The following relations hold:

$$[e_i, f_j] = \delta_{ij}\check{\alpha}_i, [h, e_i] = \alpha_i(h)e_i, \text{ and } [h, f_i] = -\alpha_i(h)f_i,$$

for any $i, j = 0, \cdots, n$ and $h \in \mathfrak{h}$. Let $\mathfrak{g}'(A)$ denote the derived subalgebra $[\mathfrak{g}(A), \mathfrak{g}(A)]$ of $\mathfrak{g}(A)$. Note that $\mathfrak{g}'(A)$ is generated by $e_i, f_i(i = 0, \cdots, n)$. Fix an element $d \in \mathfrak{h}$ such that

$$\langle \alpha_i, d \rangle = 0 \text{ for } i = 1, \cdots, \quad \langle \alpha_0, d \rangle = 1.$$

Then $\mathfrak{g}(A) = \mathfrak{g}'(A) + \mathbb{C}d$, and $\mathfrak{h}$ is spanned by $\check{\alpha}_i(i = 0, \cdots, n)$ and d.

Put

$$K = \sum_{i=0}^{n} \check{a}_i \check{\alpha}_i.$$

Then K is the central element of $\mathfrak{g}(A)$.

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Let $\Lambda_i (i = 0, \cdots, n)$ denote the fundamental weights of $\mathfrak{g}(A)$. The space $\mathfrak{h}^*$ is spanned by $\alpha_i, i = 0, \cdots, n$ and Λ_0. There is a special imaginary root $\delta \in \mathfrak{h}^*$ given by $\delta = \sum_{i=0}^{n} a_i \alpha_i$. Let $(\cdot|\cdot)$ denote the symmetric normalized bilinear form on $\mathfrak{h}$ such that:

$$\begin{cases} (\check{\alpha}_i|\check{\alpha}_j) = a_j \check{a}_j a_{ij} \quad (i,j = 0, \cdots, n) \\ (\check{\alpha}_i|d) = 0 \quad i = 1, \cdots, n \\ (\check{\alpha}_0|d) = a_0; \quad (d|d) = 0. \end{cases}$$

Let $\kappa : \mathfrak{h} \to \mathfrak{h}^*$ be the induced isomorphism from $(\cdot|\cdot)$. Then

$$\kappa(\check{\alpha}_i) = \frac{a_i}{\check{a}_i} \alpha_i; \kappa(K) = \delta, \kappa(d) = a_0 \Lambda_0.$$

We denote by $\mathring{\mathfrak{g}}$ the subalgebra of $\mathfrak{g}(A)$ generated by $e_i, f_i\ (i = 1, \cdots, n)$. Let $\mathring{\mathfrak{h}}$ be the span of $\check{\alpha}_i\ (i = 1, \cdots, n)$. Then $\mathring{\mathfrak{g}}$ is a simple Lie algebra with the Cartan subalgebra $\mathring{\mathfrak{h}}$. Let $\check{h}$ denote the dual Coxeter number of $\mathfrak{g}(A)$. We denote by $\mathring{A}$ the Cartan matrix of $\mathring{\mathfrak{g}}$.

We will use $X_N^{(r)}$ to denote the generalized Cartan matrix of affine type. When $r = 1$, $X_N^{(1)}$ is untwisted; otherwise $X_N^{(r)}$ is twisted.

The following is the table for the associated $\mathring{A}$ and $\check{h}$ when A is of untwisted affine type.

(1)

A	$A_n^{(1)}$	$B_n^{(1)}$	$C_n^{(1)}$	$D_n^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	$F_4^{(1)}$	$G_2^{(1)}$
$\mathring{A}$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$\check{h}$	n+1	2n-1	n+1	2n-2	12	18	30	9	4

In the above table, $n \geq 1$ when $A = A_n^{(1)}$, $n \geq 3$ when $A = B_n^{(1)}$, $n \geq 2$ when $A = C_n^{(1)}$, and $n \geq 4$ when $A = D_n^{(1)}$.

The following is the table for the associated $\mathring{A}$ and $\check{h}$ when A is of twisted affine type.

(2)

A	$A_{2n}^{(2)}$	$A_{2n-1}^{(2)}$	$D_{n+1}^{(2)}$	$E_6^{(2)}$	$D_4^{(3)}$
$\mathring{A}$	C_n	C_n	B_n	F_4	G_2
$\check{h}$	2n+1	2n	2n	12	6

In the above table, $n \geq 2$ when $A = A_{2n}^{(2)}$, $n \geq 3$ when $A = A_{2n-1}^{(2)}$, and $n \geq 3$ when $A = D_{n+1}^{(2)}$.

2. Fusion rings associated to Kac-Moody algebras of affine types

Let $\mathring{P} \subset \mathring{\mathfrak{h}}^*$ (resp. $\mathring{Q} \subset \mathring{\mathfrak{h}}^*$) be the weight lattice (resp. root lattice) associated to $\mathring{\mathfrak{g}}$. Let $\check{\mathring{P}}$ denote the dual lattice of $\mathring{Q}$, i.e.

$$\check{\mathring{P}} = \{\check{\mu} \in \mathring{\mathfrak{h}} \,|\, \langle \lambda, \check{\mu} \rangle \in \mathbb{Z}, \text{ for any } \lambda \in \mathring{Q}\}.$$

Similarly let $\check{\mathring{Q}}$ denote the dual lattice of $\mathring{P}$. Let $\mathring{P}^+$ (resp. $\check{\mathring{P}}^+$) be the set of dominant weights (resp. dominant coweights) of $\mathring{\mathfrak{g}}$. Let $\mathring{\Phi}$ (resp. $\mathring{\Phi}^+$) denote the set of roots (resp. positive roots) of $\mathring{\mathfrak{g}}$.

Let $\mathring{\rho}$ (resp. $\check{\mathring{\rho}}$) denote the summation of fundamental weights (resp. coweights) of $\mathfrak{g}$. Recall the symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$. It restricts to the subspace $\mathring{\mathfrak{h}}$, and it induces the isomorphism $\kappa : \mathring{\mathfrak{h}} \simeq \mathring{\mathfrak{h}}^*$.

Put

$$M = \begin{cases} \kappa(\mathring{\check{Q}}) & \text{if } A = X_N^{(1)} \text{ or } A = A_{2n}^{(2)} \\ \mathring{Q} & \text{otherwise} \end{cases} . \tag{3}$$

Let W (resp. $\mathring{W}$) denote the Weyl group of $\mathfrak{g}(A)$ (resp. $\mathring{\mathfrak{g}}$). Then W is naturally isomorphic to $\mathring{W} \ltimes M$ (cf.[**Ka**, Proposition 6.5]).

Let θ be the highest root of $\mathring{\mathfrak{g}}$. Set

$$\check{\theta} = \begin{cases} \kappa^{-1}(\theta) & \text{if } A = X_N^{(1)} \text{ or } A = A_{2n}^{(2)} \\ \text{highest coroot of } \mathring{\mathfrak{g}} & \text{otherwise.} \end{cases} \tag{4}$$

In fact when $A = X_N^{(1)}$, $\check{\theta}$ is the highest short coroot of $\mathring{\mathfrak{g}}$; when $A = A_{2n}^{(2)}$, $\check{\theta}$ is twice of the highest short coroot of $\mathring{\mathfrak{g}}$.

Given any $k \in \mathbb{N}$. We put

$$P_k = \{\lambda \in \mathring{P}^+ \,|\, \langle \lambda, \check{\theta} \rangle \leq k\}. \tag{5}$$

We also put

$$\check{P}_k = \{\check{\lambda} \in \mathring{\check{P}}^+ \,|\, \langle \check{\lambda}, \theta \rangle \leq k\},$$

(we only need it when $A = X_N^{(r)}$ with $r > 1$ and $A \neq A_{2n}^{(2)}$).

Let $\mathring{T}$ denote the torus $\mathrm{Hom}(\mathring{P}, \mathbb{C}^\times)$. We define a finite subset Σ_k of $\mathring{T}$. When $A = X_N^{(1)}$ or $A = A_{2n}^{(2)}$, we set

$$\Sigma_k = \{e^{\frac{2\pi i}{k+h}(\mathring{\rho}+\lambda|\cdot)} \in \mathring{T} \,|\, \lambda \in P_k\}. \tag{6}$$

When $A = X_N^{(r)}$ with $r > 1$ and $A \neq A_{2n}^{(2)}$, we set

$$\Sigma_k = \{e^{\frac{2\pi i}{k+\check{h}}\langle \mathring{\check{\rho}}+\check{\lambda}, \cdot \rangle} \in \mathring{T} \,|\, \check{\lambda} \in \check{P}_k\}.$$

Let $R_k(A)$ denote the commutative ring of $\mathbb{C}$-valued functions on the finite set Σ_k. We introduce an involution $*$ on the set P_k by sending λ to $-w_0(\lambda)$ where w_0 is the longest element in $\mathring{W}$.

For any $w \in \mathring{W}$, let $\ell(w)$ be the length of w. For any $\lambda \in P_k$ and $t \in \Sigma_k$, put

$$J_\lambda(t) := \sum_{w \in \mathring{W}} (-1)^{\ell(w)} w(\lambda + \mathring{\rho})(t), \tag{7}$$

and $\mathring{\chi}_\lambda(t) := \frac{J_\lambda(t)}{J_0(t)}$.

Let $\mathring{V}_\lambda$ denote the irreducible representation of $\mathring{\mathfrak{g}}$ of highest weight λ. By Weyl character formula $\mathring{\chi}_\lambda(t)$ is the trace of t on the representation $\mathring{V}_\lambda$. Therefore we get a collection of well-defined functions $\{\mathring{\chi}_\lambda \,|\, \lambda \in P_k\}$ on Σ_k.

Set $\Delta(t) = |J_0(t)|^2$. We introduce an Hermitian form $(\cdot, \cdot)$ on $R_k(A)$ as follows, for any $f, g \in R_k(A)$

$$(f, g) := \frac{1}{|\mathring{P}/(k+\check{h})M|} \sum_{t \in \Sigma_k} f(t)\overline{g(t)}\Delta(t). \tag{8}$$

THEOREM 2.1. (1) *The Hermitian form $(\cdot, \cdot)$ on $R_k(A)$ is an inner product.*

(2) *The set of functions $\{\mathring{\chi}_\lambda \,|\, \lambda \in P_k\}$ is an orthonormal basis of the ring $R_k(A)$ with respect to the inner product $(\cdot, \cdot)$.*

As a ring of functions on a finite set, $R_k(A)$ is semisimple. For any $\lambda, \mu \in P_k$, there exist unique coefficients $c_{\lambda\mu}^{\nu}$ such that

$$\mathring{\chi}_\lambda \cdot \mathring{\chi}_\mu = \sum_{\nu \in P_k} c_{\lambda\mu}^{\nu} \mathring{\chi}_\nu,$$

where $c_{\lambda\mu}^{\nu} \in \mathbb{C}$. Observe that $\mathring{\chi}_{\lambda^*}(t) = \overline{\mathring{\chi}_\lambda(t)}$ for any $\lambda \in P_k$ and $t \in \Sigma_k$. Then an immediate consequence of Theorem 2.1 is that the coefficients $c_{\lambda\mu}^{\nu}$ can be computed by the following formula

$$c_{\lambda\mu}^{\nu} = \frac{1}{|\mathring{P}/(k+\check{h})M|} \sum_{t \in \Sigma_k} \mathring{\chi}_\lambda(t) \mathring{\chi}_\mu(t) \mathring{\chi}_{\nu^*}(t) \Delta(t). \tag{9}$$

We call the ring $R_k(A)$ together with the basis $\{\mathring{\chi}_\lambda \,|\, \lambda \in P_k\}$ the fusion ring associated to the Kac-Moody algebra $\mathfrak{g}(A)$, and we call $c_{\lambda\mu}^{\nu}$ the fusion coefficients of $R_k(A)$.

The following theorem asserts the integrability of fusion coefficients, and the stabilization when the level is sufficient large.

THEOREM 2.2. *For any $\lambda, \mu, \nu \in P_k$, we have*

(1) *The coefficients $c_{\lambda\mu}^{\nu} \in \mathbb{Z}$.*
(2) *If $\langle \lambda + \mu + \nu, \check{\theta} \rangle \leq 2k$, then*

$$c_{\lambda\mu}^{\nu} = \dim \operatorname{Hom}_{\mathring{\mathfrak{g}}}(\mathring{V}_\nu, \mathring{V}_\lambda \otimes \mathring{V}_\mu).$$

The following theorem is an incarnation of [**Ho2**, Theorem 1.6] in terms of fusion rings.

THEOREM 2.3. *The fusion rings $R_{2k+1}(A_{2n}^{(2)})$ and $R_k(C_n^{(1)})$ are isomorphic.*

Theorem 2.1, Theorem 2.2 and Theorem 2.3 are basically the consequence of Verlinde formula for dimension of conformal blocks (cf.[**Be**]), and the Verlinde formula for the trace of diagram automorphism on conformal blocks [**Ho2**] by the author. In the rest of note we will explain uniformly how they are deduced.

When $A = X_N^{(1)}$, the theory of fusion rings and conformal blocks can be dated back to [**Fa**, **Be**, **TUY**, **V**]. The coefficients $c_{\lambda\mu}^{\nu}$ are always nonnegative, since they can be interpreted as the dimension of conformal blocks.

When $A = X_N^{(r)}$ with $r > 1$, the corresponding theory of fusion rings and its relation with usual conformal blocks is essentially explained in [**Ho2**], although the role of twisted affine Lie algebras was not clearly clarified there. We expect that the fusion coefficients $c_{\lambda\mu}^{\nu}$ are all nonnegative. For example by Theorem 2.3 the fusion coefficients for $A_{2n}^{(2)}$ are always nonnegative when the level is odd. We expect that there is an appropriate theory of twisted conformal blocks whose dimensions would correspond to the fusion coefficients for twisted affine Lie algebras. We also hope that there will be a theory of fusion category for twisted affine Lie algebra in the flavor of [**Fi**].

3. Twining formula and Verlinde formula

3.1. Twining formula for tensor invariant space. For each $\mathfrak{g}(A)$ with $A = X_N^{(r)}$, we attach a unique pair $(\dot{\mathfrak{g}}, \sigma)$ where $\dot{\mathfrak{g}}$ is a simple Lie algebra and σ is a diagram automorphism on $\dot{\mathfrak{g}}$ (possibly trivial) of order r. Let $\mathring{A}$ (resp. $\dot{A}$) denote the Cartan matrix of $\mathring{\mathfrak{g}}$ (resp. $\dot{\mathfrak{g}}$). The Cartan matrix $\dot{A}$ is attached as follows. If A is of untwisted affine type, then we take $\dot{A} = \mathring{A}$; if A is of twisted affine type, we have the following table:

(10)

A	$A_{2n}^{(2)}$	$A_{2n-1}^{(2)}$	$D_{n+1}^{(2)}$	$E_6^{(2)}$	$D_4^{(3)}$
$\mathring{A}$	C_n	C_n	B_n	F_4	G_2
$\dot{A}$	A_{2n}	D_{n+1}	A_{2n-1}	E_6	D_4

The simple Lie algebra $\mathring{\mathfrak{g}}$ is the orbit Lie algebra of the pair $(\dot{\mathfrak{g}}, \sigma)$ in the sense of [**Ho2**, Section 2.1].

Let $\mathring{I}$ (resp. $\dot{I}$) denote the set of vertices of Dynkin diagram of $\mathring{\mathfrak{g}}$ (resp. $\dot{\mathfrak{g}}$). We will denote by $\dot{P}$ (resp. $\dot{Q}$) the weight lattice (resp. root lattice) of $\dot{\mathfrak{g}}$, and we denote by $\dot{P}^+$ the set of dominant weights. Let $\{\dot{\omega}_j \mid j \in \dot{I}\}$ denote the fundamental weights of $\dot{\mathfrak{g}}$. There is a bijection $p : \mathring{I} \simeq \dot{I}/\sigma$, where $\dot{I}/\sigma$ denotes the set of σ-orbits on $\dot{I}$. Moreover there is a bijection $\iota : \mathring{P} \simeq \dot{P}^\sigma$ and a projection $\breve{\iota} : \check{\dot{Q}} \to \check{\mathring{Q}}$ such that

(1) $\iota(\mathring{\omega}_i) = \sum_{j \in p(i)} \dot{\omega}_i$ for any $i \in \mathring{I}$.

(2) $\langle \lambda, \breve{\iota}(\check{\beta}) \rangle = \langle \iota(\lambda), \beta \rangle$, for any $\lambda \in \mathring{P}$ and $\check{\beta} \in \check{\dot{Q}}$.

For any $\lambda \in \mathring{P}^+$, let $\dot{V}_\lambda$ denote the irreducible representation of $\dot{\mathfrak{g}}$ of highest weight $\iota(\lambda) \in \dot{P}^+$. There is a unique operator $\sigma : \dot{V}_\lambda \to \dot{V}_\lambda$ such that

(1) $\sigma(x \cdot v) = \sigma(x) \cdot \sigma(v)$ for any $x \in \dot{\mathfrak{g}}$ and $v \in \dot{V}$,

(2) $\sigma(v_\lambda) = v_\lambda$ where v_λ is the highest weight vector of $\dot{V}_\lambda$.

For any tuple $\vec{\lambda} = (\lambda_1, \cdots, \lambda_m)$ of elements in $\mathring{P}^+$, let $\dot{V}_{\vec{\lambda}}^{\dot{\mathfrak{g}}}$ (resp. $\mathring{V}_{\vec{\lambda}}^{\mathring{\mathfrak{g}}}$) denote the tensor invariant space

$$(\dot{V}_{\lambda_1} \otimes \cdots \otimes \dot{V}_{\lambda_m})^{\dot{\mathfrak{g}}} \text{ (resp. } (\mathring{V}_{\lambda_1} \otimes \cdots \otimes \mathring{V}_{\lambda_m})^{\mathring{\mathfrak{g}}}).$$

From each operator σ on $\dot{V}_{\lambda_i}$, there exists an operator σ (we still use the same notation) acting diagonally on $\dot{V}_{\vec{\lambda}}^{\dot{\mathfrak{g}}}$. The following twining formula was proved in [**HS**, Theorem 1.1].

THEOREM 3.1. *We have the equality* $\operatorname{tr}(\sigma | \dot{V}_{\vec{\lambda}}^{\dot{\mathfrak{g}}}) = \dim \mathring{V}_{\vec{\lambda}}^{\mathring{\mathfrak{g}}}$, *where* $\operatorname{tr}(\sigma | \dot{V}_{\vec{\lambda}}^{\dot{\mathfrak{g}}})$ *is the trace of* σ *on* $\dot{V}_{\vec{\lambda}}^{\dot{\mathfrak{g}}}$.

The twining formula for the weight spaces between representations of $\dot{\mathfrak{g}}$ and $\mathring{\mathfrak{g}}$ was first discovered by Jantzen (cf. [**Ho1**, **Ja**]). Theorem 3.1 can actually be deduced from Jantzen formula (cf.[**Ho2**, Section 5.1]).

3.2. Verlinde formula. Let $\mathbb{C}((t))$ be the field of Laurent series. Let $\hat{\mathcal{L}}(\dot{\mathfrak{g}})$ be the associated affine Lie algebra $\mathfrak{g}((t)) \oplus \mathbb{C}K \oplus \mathbb{C}d$ (cf.[**Ka**, §7]). The diagram automorphism σ still acts on $\hat{\mathcal{L}}(\dot{\mathfrak{g}})$. Note that if A is of untwisted affine type, then $\mathfrak{g}(A) \simeq \hat{\mathcal{L}}(\mathfrak{g})$. If A is of twisted affine type, then the Kac-Moody algebra $\mathfrak{g}(A)$ is the orbit Lie algebra of $(\hat{\mathcal{L}}(\dot{\mathfrak{g}}), \sigma)$, and in fact there is also a twining formula for the weight spaces between representations of $\hat{\mathcal{L}}(\dot{\mathfrak{g}})$ and $\mathfrak{g}(A)$, which is due to Fuchs-Schellekens-Schweigert[**FSS**].

Let $\check{\dot{\theta}}$ denote the highest short coroot of $\dot{\mathfrak{g}}$. Let $\dot{P}_k$ denote the subset of $\dot{P}^+$ consisting of $\lambda \in \dot{P}^+$ such that $\langle \lambda, \check{\dot{\theta}} \rangle \leq k$. For any $\lambda \in \dot{P}_k$, let $\dot{\mathcal{H}}_{\lambda + k\dot{\Lambda}_0}$ denote the irreducible integrable representation of $\hat{\mathcal{L}}(\dot{\mathfrak{g}})$ of highest weight $\lambda + k\dot{\Lambda}_0$, where $\dot{\Lambda}_0$

is the fundamental weight of $\hat{\mathcal{L}}(\dot{\mathfrak{g}})$ corresponding to the vertex 0 in the extended Dynkin diagram of $\dot{\mathfrak{g}}$.

LEMMA 3.2. *The isomorphism $\iota : \mathring{P} \simeq \dot{P}^\sigma$ restricts to the bijection $\iota : P_k \simeq (\dot{P}_k)^\sigma$.*

PROOF. It is enough to show that for any $\lambda \in P_k$, $\iota(\lambda) \in \dot{P}_k$. By [**Ho2**, Lemma 2.1], $\check{\iota}(\dot{\check{\theta}}) = \check{\theta}$. For any $\lambda \in P_k$, we have

$$\langle \iota(\lambda), \dot{\check{\theta}} \rangle = \langle \lambda, \check{\iota}(\dot{\check{\theta}}) \rangle = \langle \lambda, \check{\theta} \rangle \leq k.$$

Hence $\iota(\lambda) \in \dot{P}_k$. □

Given an m-pointed smooth projective curve $(C, \vec{p})$ over $\mathbb{C}$, on each point p_i we associate a dominant weight $\lambda_i \in \dot{P}_k$ and an irreducible integral representation $\dot{\mathcal{H}}_{\lambda_i + k\dot{\Lambda}}$ an affine Lie algebra $\hat{\mathcal{L}}(\dot{\mathfrak{g}})$ depending on the point p_i. Let $\dot{\mathfrak{g}}(C\backslash\vec{p})$ be the space of $\dot{\mathfrak{g}}$-valued regular functions on $C\backslash\vec{p}$. The space $\dot{\mathfrak{g}}(C\backslash\vec{p})$ is naturally a Lie algebra induced from $\dot{\mathfrak{g}}$. The Lie algebra $\dot{\mathfrak{g}}(C\backslash\vec{p})$ acts on

$$\dot{\mathcal{H}}_{\vec{\lambda}} := \dot{\mathcal{H}}_{\lambda_1 + k\dot{\Lambda}_0} \otimes \cdots \otimes \dot{\mathcal{H}}_{\lambda_m + k\dot{\Lambda}_0}$$

naturally. The space $\dot{V}_{\vec{\lambda}}(C, \vec{p})$ of conformal blocks associated to $\vec{p}$ and $\vec{\lambda}$ is defined as the coinvariant space of $\dot{\mathcal{H}}_{\vec{\lambda}}$ with respect to the action of $\dot{\mathfrak{g}}(C\backslash\vec{p})$:

$$\dot{V}_{\vec{\lambda}}(C, \vec{p}) := \dot{\mathcal{H}}_{\vec{\lambda}} / \dot{\mathfrak{g}}(C\backslash\vec{p}) \dot{\mathcal{H}}_{\vec{\lambda}}.$$

The trace $\mathrm{tr}(\sigma | \dot{V}_{\vec{\lambda}}(C, \vec{p}))$ is independent of the choice of $\vec{p}$ (cf.[**Ho2**, Section 3]). For convenience we write $\mathrm{tr}(\sigma | \dot{V}_{\vec{\lambda}}(C))$ by ignoring $\vec{p}$.

LEMMA 3.3. *We attach $\lambda, \mu \in \dot{P}_k$ to two distinct points on the projective line $\mathbb{P}^1$. Then we have*

$$\mathrm{tr}(\sigma | \dot{V}_{\lambda,\mu}(\mathbb{P}^1)) = \delta_{\lambda,\mu^*}.$$

PROOF. cf.[**Ho2**, Lemma 3.9]. □

Let $\dot{\check{h}}$ denote the dual Coxeter number of $\hat{\mathcal{L}}(\dot{\mathfrak{g}})$. By comparing Table (1), (2) and (10), we can see that $\dot{\check{h}} = \check{h}$.

LEMMA 3.4. *If $A = X_N^{(1)}$ or $A = A_{2n}^{(2)}$, then*

$$\kappa(\mathring{\check{Q}}) = \begin{cases} Q_l & A = X_N^{(1)} \\ \frac{1}{2} Q_l & A = A_{2n}^{(2)} \end{cases}.$$

PROOF. It follows from the discussions in [**Ka**, §6.5]. □

Recall the torus $\mathring{T} = \mathrm{Hom}(\mathring{P}, \mathbb{C})$. Set

$$T_k := \{ t \in \mathring{T} \,|\, \alpha(t) = 1, \forall \alpha \in (k + \check{h})M \}.$$

Then T_k is a finite subgroup of $\mathring{T}$. An element $t \in \mathring{T}$ is regular if $\alpha(t) \neq 1$ for any $\alpha \in \mathring{\Phi}$. Equivalently an element $t \in \mathring{T}$ is regular if and only if the stabilizer group of $\mathring{W}$ at t is trivial. Let T_k^{reg} denote the set of regular elements in T_k.

The following is the general Verlinde formula for the trace of σ on the space of conformal blocks.

THEOREM 3.5. *Let $(C, \vec{p})$ be an m-pointed smooth projective curve of genus g. Given a tuple $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ of elements in P_k. We have the following formula*

$$\operatorname{tr}(\sigma|\dot{V}_{\vec{\lambda}}(C)) = |T_k|^{g-1} \sum_{t \in T_k^{\mathrm{reg}}/\mathring{W}} \mathring{\chi}_{\lambda_1}(t) \cdots \mathring{\chi}_{\lambda_m}(t) \Delta(t)^{1-g}. \tag{11}$$

PROOF. When $A = X_N^{(1)}$, it is just the usual Verlinde formula(cf.[**Be**]). When $A = X_N^{(r)}$ with $r > 1$, it is the main theorem of [**Ho2**]. Lemma 3.4 ensures the data in the formulas of [**Be**, **Ho2**] matches with the note. □

In particular if $C = \mathbb{P}^1$ and $\vec{\lambda} = (\lambda, \mu, \nu)$ with $\lambda, \mu, \nu \in P_k$, we have

$$\operatorname{tr}(\sigma|\dot{V}_{\vec{\lambda}}(C)) = \frac{1}{|T_k|} \sum_{t \in T_k^{\mathrm{reg}}/\mathring{W}} \mathring{\chi}_{\lambda}(t) \mathring{\chi}_{\mu}(t) \mathring{\chi}_{\nu}(t) \Delta(t). \tag{12}$$

Let W_k denote the affine Weyl group $\mathring{W} \ltimes (k + \check{h})M$. We define the dotted action of W_k on $\mathring{P}$,

$$w \star \lambda = w(\lambda + \mathring{\rho}) - \mathring{\rho}, \text{ for any } w \in W_k.$$

Let $W_k^{\dagger}$ denote the set consisting of the minimal representatives of the left cosets of $\mathring{W}$ in W_k. For any $\lambda \in P_k$, $w \star \lambda \in \mathring{P}^{+}$ if and only if $w \in W_k^{\dagger}$.

The following is a general version of Kac-Walton formula.

THEOREM 3.6. *For any $\lambda, \mu, \nu \in P_k$, we have*

$$\operatorname{tr}(\sigma|\dot{V}_{\lambda,\mu,\nu}(\mathbb{P}^1)) = \sum_{w \in W_k^{\dagger}} (-1)^{\ell(w)} \dim \mathring{V}^{\mathring{\mathfrak{g}}}_{\lambda,\mu,w\star\nu}.$$

PROOF. When $A = X_N^{(1)}$ it is just the usual Kac-Walton formula (cf.[**Ka**, Ex.13.35]). When $A = X_N^{(r)}$ with $r > 1$, it follows from [**Ho2**, Theorem 5.11] and Theorem 3.1. □

4. Proofs

4.1. Proof of Theorem 2.1.

4.1.1. *Inner product.*

LEMMA 4.1. *We have the following equality*

$$\check{h} = \langle \mathring{\rho}, \check{\theta} \rangle + 1$$

PROOF. When $A = X_N^{(1)}$, $\check{\theta}$ is the highest short coroot of $\mathring{\mathfrak{g}}$. When $A = A_{2n}^{(2)}$, $\check{\theta}$ is twice of the highest short coroot of $\mathring{\mathfrak{g}}$. Then it is easy to check that $\check{h} = (\mathring{\rho}, \check{\theta}) + 1$ by comparing the numerical labels in the tables of [**Ka**, §4.8] and Tables (1) (2).

When $A = X_N^{(r)}$ with $r > 1$ and $A \neq A_{2n}^{(2)}$, $\langle \mathring{\rho}, \check{\theta} \rangle + 1$ is the Coxeter number of dual root system of $\mathring{\mathfrak{g}}$. By comparing the table in [**Ka**, §6.1], it is also easy to see the equality. □

To prove the Hermitian form defined in (8) is an inner product, it is enough to show that $\Delta(t) > 0$ for any $t \in \Sigma_k$. By Weyl denominator formula

$$\Delta(t) = \prod_{\alpha \in \mathring{\Phi}} (1 - \alpha(t)) = |\prod_{\alpha \in \mathring{\Phi}^{+}} (1 - \alpha(t))|^2.$$

When $A = X_N^{(1)}$ or $A = A_{2n}^{(2)}$, for any $\lambda \in P_k$ and $\alpha \in \mathring{\Phi}^+$, we have $0 \leq (\lambda|\alpha) \leq k$. By Lemma 4.1, $(\mathring{\rho}|\theta) = \langle \mathring{\rho}, \check{\theta} \rangle = \check{h} - 1$. Since θ is the highest root of $\mathring{\mathfrak{g}}$, for any $\alpha \in \mathring{\Phi}^+$ we have

$$0 < (\mathring{\rho}|\alpha) \leq (\mathring{\rho}|\theta) = \check{h} - 1.$$

In particular

$$0 < (\lambda + \mathring{\rho}|\alpha) \leq k + \check{h} - 1.$$

It shows that $\alpha(t)$ can not be equal to 1. Hence $\Delta(t) > 0$ for any $t \in \Sigma_k$.

When $A = X_N^{(r)}$ with $r > 1$ and $A \neq A_{2n}^{(2)}$, for any $\check{\lambda} \in \check{P}_k$, we have $0 \leq \langle \alpha, \check{\lambda} \rangle \leq k$. By Lemma 4.1, for any $\alpha \in \mathring{\Phi}^+$, we always have

$$0 < \langle \alpha, \mathring{\check{\rho}} + \check{\lambda} \rangle < k + \check{h}.$$

Hence in this case, we also have $\Delta(t) > 0$ for any $t \in \Sigma_k$.

4.1.2. *Orthonormal basis.* Recall that T_k is the group consisting of $t \in \mathring{T}$ such that $\alpha(t) = 1$ for any $\alpha \in (k + \check{h})M$. The group T_k is naturally isomorphic to

$$\begin{cases} \mathring{P}/(k+\check{h})M & \text{if } A = X_N^{(1)} \text{ or } A = A_{2n}^{(2)} \\ \mathring{\check{P}}/(k+\check{h})\mathring{Q} & \text{otherwise} \end{cases}. \tag{13}$$

LEMMA 4.2. $|T_k| = |\mathring{P}/(k + \check{h})M|$.

PROOF. When $A = X_N^{(1)}$ or $A = A_{2n}^{(2)}$, it simply follows from the natural isomorphism (13).

When $A = X_N^{(r)}$ with $r > 1$ and $A \neq A_{2n}^{(2)}$, $\mathring{\check{P}}/(k + \check{h})\mathring{\check{Q}}$ is Pontryagin dual to $\mathring{P}/(k + \check{h})\mathring{Q}$. Then the lemma follows from the natural isomorphism (13). $\square$

We recall the finite set Σ_k in $\mathring{T}$. It is clear that Σ_k is a subset of T_k^{reg}.

When $A = X_N^{(1)}$ or $A = A_{2n}^{(2)}$, let $\eta_1 : P_k \to \Sigma_k$ be the map by assigning $\lambda \in P_k$ to $t_\lambda := e^{\frac{2\pi i}{k+\check{h}}(\mathring{\rho}+\lambda|\cdot)} \in \Sigma_k$. When $A = X_N^{(r)}$ with $r > 1$ and $A \neq A_{2n}^{(2)}$, let $\eta_2 : \check{P}_k \to \Sigma_k$ be the map assigning $\check{\lambda} \in \check{P}_k$ to $t_{\check{\lambda}} := e^{\frac{2\pi i}{k+\check{h}}\langle \mathring{\check{\rho}}+\check{\lambda},\cdot\rangle}$.

LEMMA 4.3. *The maps η_1, η_2 defined above are bijections.*

PROOF. We first assume that $A = X_N^{(1)}$ or $A = A_{2n}^{(2)}$. It is enough to show that η_1 is injective, equivalent it is enough to show that $i_1 : P_k \to \mathring{P}/(k+\check{h})M$ the map given by

$$\lambda \mapsto \lambda + \mathring{\rho} \mod (k + \check{h})M$$

is injective. For any $\lambda \in (k + \check{h})M$ and $w \in \mathring{W}$, we always have

$$\langle \lambda, w(\check{\theta}) \rangle \in (k + \check{h})\mathbb{Z}. \tag{14}$$

For any $\lambda_1, \lambda_2 \in P_k$, we have $0 \leq \langle \lambda_i, \check{\theta} \rangle \leq k$. It also follows that for any $w \in \mathring{W}$,

$$-k \leq \langle \lambda_i, w(\check{\theta}) \rangle \leq k. \tag{15}$$

If $i(\lambda_1) = i(\lambda_2)$, equivalently $\lambda_1 - \lambda_2 \in (k + \check{h})M$, then by (14) and (15) we have

$$\langle \lambda_1 - \lambda_2, w(\check{\theta}) \rangle = 0,$$

for any $w \in \mathring{W}$. Since $\{w(\check{\theta}) \,|\, w \in \mathring{W}\}$ spans $\mathring{\mathfrak{h}}$, it follows that $\lambda_1 = \lambda_2$. It proves the injectivity of i_1.

When $A = X_N^{(r)}$ with $r > 1$ and $A \neq A_{2n}^{(2)}$, we consider the map $i_2 : \check{P}_k \to \check{\mathring{P}}/(k+\check{h})\check{\mathring{Q}}$ given by

$$\check{\lambda} \mapsto \check{\lambda} + \check{\mathring{\rho}} \mod (k+\check{h})\check{\mathring{Q}}.$$

In this case the lemma can be proved similarly. □

LEMMA 4.4. *When $A = X_N^{(r)}$ with $r > 1$ and $A \neq A_{2n}^{(2)}$, we have $|P_k| = |\check{P}_k|$.*

PROOF. When $A \neq A_{2n-1}^{(2)}$ and $A \neq D_{n+1}^{(2)}$, it is clear that $|P_k| = |\check{P}_k|$ since the dual Lie algebra of $\mathring{\mathfrak{g}}$ is isomorphic to $\mathring{\mathfrak{g}}$.

When $A = A_{2n-1}^{(2)}$ or $A = D_{n+1}^{(2)}$, we define a map $\epsilon : \mathring{P} \to \check{\mathring{P}}$ given by $\omega_i \mapsto \check{\omega}_{\ell-i}$ where ℓ is the rank of $\mathring{\mathfrak{g}}$. Note that ϵ induces the dual map $\check{\epsilon} : \mathring{Q} \to \check{\mathring{Q}}$. Observe that $\check{\epsilon}(\theta) = \check{\theta}$, where $\check{\theta}$ is the highest coroot of $\mathring{\mathfrak{g}}$. It is now clear that ϵ defines a bijection $P_k \simeq \check{P}_k$. Hence the lemma is proved. □

COROLLARY 4.5. *For any generalized Cartan matrix A of affine type, we have $|\Sigma_k| = |P_k|$.*

PROOF. It follows from Lemma 4.2 and Lemma 4.3. □

LEMMA 4.6. *Σ_k is the fundamental set of T_k^{reg} with respect to the action of the Weyl group $\mathring{W}$, i.e. for any $t \in T_k^{\mathrm{reg}}$ there exists a unique element $t_0 \in \Sigma_k$ and a unique $w \in \mathring{W}$ such that $w(t_0) = t$.*

PROOF. By Lemma 4.3, we are reduced to consider the action of the Weyl group $\mathring{W}$ on $\mathring{P}/(k+\check{h})M$.

When $A = X_N^{(1)}$ or $A = A_{2n}^{(2)}$, it follows from the fact that the set $\{\mathring{\rho}+\lambda \,|\, \lambda \in P_k\}$ is exactly those integral weights sitting in the interior of fundamental alcove with respect to the action of the affine Weyl group $\mathring{W} \ltimes (k+\check{h})M$ on $\mathring{P} \otimes \mathbb{R}$.

When $A = X_N^{(r)}$ with $r > 1$ and $A \neq A_{2n}^{(2)}$, it follows from the fact that the set $\{\check{\mathring{\rho}} + \check{\lambda} \,|\, \lambda \in \check{P}_k\}$ is those integral coweights sitting in the interior of fundamental alcove with respect to the action of the affine Weyl group $\mathring{W} \ltimes (k+\check{h})\check{\mathring{Q}}$ on $\check{\mathring{P}} \otimes \mathbb{R}$. □

From this lemma, we immediately get $\Sigma_k = T_k^{\mathrm{reg}}/\mathring{W}$. By Theorem 3.5, Lemma 4.2 and Lemma 4.6, Lemma 3.3 is equivalent to

$$\frac{1}{|\mathring{P}/(k+\check{h})M|} \sum_{t \in \Sigma_k} \mathring{\chi}_\lambda(t) \mathring{\chi}_{\mu^*}(t) \Delta(t) = \delta_{\lambda\mu}, \tag{16}$$

for any $\lambda, \mu \in P_k$. Recall the definition of the Hermitian form $(\cdot,\cdot)$ in (8), the formula (16) implies that $\{\mathring{\chi}_\lambda \,|\, \lambda \in P_k\}$ is an orthonormal subset of $R_k(A)$.

In the end in view of Corollary 4.5, the dimension $R_k(A)$ is equal to the cardinality of P_k. It follows that $\{\mathring{\chi}_\lambda \,|\, \lambda \in P_k\}$ is indeed an orthonormal basis of $R_k(A)$ with respect to the inner product $(\cdot,\cdot)$.

4.2. Proof of Theorem 2.2. From the proof of Theorem 2.1, the Verlinde formula (Theorem 3.5) implies that

$$c_{\lambda\mu}^{\nu} = \mathrm{tr}(\sigma | \dot{V}^{\dot{\mathfrak{g}}}_{\lambda,\mu,\nu^*}), \tag{17}$$

for any $\lambda, \mu, \nu \in P_k$. By Kac-Walton formula (Theorem 3.6) we have

$$c^{\nu}_{\lambda\mu} = \sum_{w \in W^{\dagger}_k} (-1)^{\ell(w)} \dim \mathring{V}^{\mathring{\mathfrak{g}}}_{\lambda,\mu,w(\mu^*)}. \tag{18}$$

It follows that for any $\lambda, \mu, \nu \in P_k$, $c^{\nu}_{\lambda\mu} \in \mathbb{Z}$.

We now proceed to prove the stabilization of fusion coefficients. We first observe that $\langle \lambda^*, \check{\theta} \rangle = \langle \lambda, \check{\theta} \rangle$ for any $\lambda \in \mathring{P}$, since $\check{\theta}^* = \check{\theta}$. Hence the condition $\langle \lambda+\mu+\nu, \check{\theta} \rangle \leq 2k$ is equivalent to the condition $\langle \lambda+\mu+\nu^*, \check{\theta} \rangle \leq 2k$.

When $A = X^{(1)}_N$, the stabilization follows from [**Be**, Proposition 4.3].

When $A = X^{(r)}_N$ with $r > 1$, $\dot{\mathfrak{g}}$ is simply-laced, and λ, μ, ν satisfies the following inequality

$$\langle \iota(\lambda) + \iota(\mu) + \iota(\nu)^*, \check{\dot{\theta}} \rangle \leq 2k.$$

Again in view of [**Be**, Proposition 4.3] we have

$$\dot{V}_{\lambda,\mu,\nu^*}(\mathbb{P}^1) \simeq (\dot{V}_\lambda \otimes \dot{V}_\mu \otimes \dot{V}_{\nu^*})_{\dot{\mathfrak{g}}}.$$

In particular it follows that

$$\mathrm{tr}(\sigma | \dot{V}_{\lambda,\mu,\nu^*}(\mathbb{P}^1)) = \mathrm{tr}(\sigma | (\dot{V}_\lambda \otimes \dot{V}_\mu \otimes \dot{V}_{\nu^*})_{\dot{\mathfrak{g}}}) = \dim(\mathring{V}_\lambda \otimes \mathring{V}_\mu \otimes \mathring{V}_{\nu^*})_{\mathring{\mathfrak{g}}},$$

where the second equality follows from Theorem 3.1, since the tensor invariant space and tensor co-invariant space of $\dot{\mathfrak{g}}$ and $\mathring{\mathfrak{g}}$ are naturally isomorphic.

4.3. Proof of Theorem 2.3. Let $A_1 = A^{(2)}_{2n}$ and $A_2 = C^{(1)}_n$. The underlying simple Lie algebras $\mathring{\mathfrak{g}}_1$ and $\mathring{\mathfrak{g}}_2$ are both isomorphic to the simple Lie algebra of type C_n. We use notation $\mathring{\mathfrak{g}}$ for type C_n simple Lie algebra to identity $\mathring{\mathfrak{g}}_1$ and $\mathring{\mathfrak{g}}_2$. Let M_1 (resp. M_2) be the lattice in $\mathring{\mathfrak{h}}^*$ associated to the generalized Cartan matrix A_1 (resp. A_2) of level $2k+1$ (resp. k). Then we have

$$M_1 = (2k+1+2n+1)\frac{1}{2}\mathring{Q}_l = (k+n+1)\mathring{Q}_l = (k+n+1)\mathring{Q}_l = M_2.$$

Let $(\cdot|\cdot)_i$ be the inner product on $\mathring{\mathfrak{h}}$ associated to A_i for each $i = 1, 2$. Note that $(\cdot|\cdot)_1 = 2(\cdot|\cdot)$. Let $P_k(A_i)$ be the finite set of dominant weights associated to A_i as in (5) for each $i = 1, 2$. In particular we have

$$P_{2k+1}(A_1) = \{\lambda \in \mathring{P} \,|\, (\lambda|\theta)_1 \leq 2k+1\}$$

and

$$P_k(A_2) = \{\lambda \in \mathring{P} \,|\, (\lambda|\theta)_2 \leq k\}.$$

It is now not difficult to see that

$$P_{2k+1}(A_1) = P_k(A_2).$$

We set $P_k := P_k(A_2) = P_{2k+1}(A_1)$. Then the fusion ring $R_{2k+1}(A_1)$ and $R_k(A_2)$ have bases indexed by the same set P_k.

Let $\Sigma_k(A_i)$ be the finite set of $\mathring{T}$ defined as in (6) for each $i = 1, 2$. We observe that

$$\Sigma_{2k+1}(A_1) = \Sigma_k(A_2),$$

since for any $\lambda \in P_k$ we have

$$e^{\frac{2\pi i}{2k+1+2n+1}(\lambda+\mathring{\rho}|\cdot)_1} = e^{\frac{2\pi i}{k+n+1}(\lambda+\mathring{\rho}|\cdot)_2}.$$

From Formula (9), we see that the fusion rings $R_{2k+1}(A_1)$ and $R_k(A_2)$ are indeed isomorphic.

5. Modular S-matrix

The relation between the fusion ring and modular S-matrix is well-known when the genearlized Cartan matrix is of untwisted affine type, see details in [**Ka**, §13.8, Ex.13.34,13.35]. In the case of twisted affine case, we use the fusion rings that we have just defined to relate to the S-matrix for twisted affine algebras (cf.[**Ka**, §13.9]). For convenience we still treat all generalized Cartan matrices of affine type generally.

We first assume $A = X_N^{(r)}$ with $r > 1$ but $A \neq A_{2n}^{(2)}$. For any A, we attach the adjacent Cartan matrix A' as follows:

(19)

A	$A_{2n-1}^{(2)}$	$D_{n+1}^{(2)}$	$E_6^{(2)}$	$D_4^{(3)}$
A'	$D_{n+1}^{(2)}$	$A_{2n+1}^{(2)}$	$E_6^{(2)}$	$D_4^{(3)}$

.

We define a linear isomorphism $\epsilon : \mathring{\mathfrak{h}}'^* \simeq \mathring{\mathfrak{h}}$ such that

(1) If $A = A_{2n-1}^{(2)}$ or $A = D_{n+1}^{(2)}$, then $\epsilon(\omega_i') = \check{\omega}_{n-i}$ for any $i = 1, \cdots, n$;
(2) otherwise $\epsilon(\omega_i') = \check{\omega}_i$ for any $i = 1, \cdots, n$,

where $\check{\omega}_i$ denotes the fundamental coweight of $\mathring{\mathfrak{g}}$.

Let $\check{\theta}'$ be an element in the coroot lattice of $\mathring{\mathfrak{g}}'$ defined similarly as $\check{\theta}$ in (4), and let P_k' be similarly defined as P_k in (5).

LEMMA 5.1. *The map ϵ restricts to a bijection $\epsilon : P_k' \simeq \check{P}_k$.*

PROOF. The root system of $\mathring{\mathfrak{g}}'$ is dual to $\mathring{\mathfrak{g}}$. The lemma is clear by looking at the Dynkin diagram of $\mathring{\mathfrak{g}}$ and $\mathring{\mathfrak{g}}'$. □

We now identify $\mathring{\mathfrak{h}}$ and $\mathring{\mathfrak{h}}'$ via the isomorphism $\varsigma := \epsilon^* \circ \kappa$ from $\mathring{\mathfrak{h}}$ to $\mathring{\mathfrak{h}}'$, where ϵ^* is the dual operator of $\epsilon : \mathring{\mathfrak{h}}'^* \simeq \mathring{\mathfrak{h}}$.

When $A = X_N^{(1)}$ or $A = A_{2n}^{(2)}$, we take $A' = A$ and take ς to be the identity on $\mathring{\mathfrak{h}}$.

When $A = X_N^{(r)}$ with $r > 1$ and $A \neq A_{2n}^{(2)}$, for any $\mu' \in P_k$, let $t_{\mu'}$ denote the element in Σ_k associated to $\epsilon(\mu') \in \check{P}_k$. When $A = X_N^{(1)}$ or $A = A_{2n}^{(2)}$, for any $\mu' \in P_k' = P_k$, we denote by $t_{\mu'}$ the element in Σ associated to $\mu' \in P_k$. Put

$$S_{\lambda,\mu'} = i^{|\mathring{\Phi}^+|} |\mathring{P}/(k+\check{h})M|^{-1/2} J_\lambda(t_{\mu'}), \tag{20}$$

where i is the imaginary unit.

The matrix $S = (S_{\lambda,\mu'})$ is indexed by $P_k \times P_k'$. We call it the modular matrix from A to A'. Let S^t (resp. $\bar{S}^t$) be the transpose (resp. conjugate transpose) of S. Then S^t (resp. $\bar{S}^t$) is a matrix indexed by $P_k' \times P_k$.

LEMMA 5.2. *$S\bar{S}^t$ is the identify matrix indexed by $P_k \times P_k$.*

PROOF. Note that the number of positive roots of type B_n is equal to the number of positive roots of type C_n (cf.[**Hu**, §12.2, Table 1]). It is easy to see that $i^{|\mathring{\Phi}^+|} = i^{|\mathring{\Phi}'^+|}$ for any A, where $\mathring{\Phi}'^+$ is the set of positive roots of $\mathring{\mathfrak{g}}'$. Hence the lemma is equivalent to that $\{\mathring{\chi}_\lambda \,|\, \lambda \in P_k\}$ is an orthonormal basis of $R_k(A)$ with respect to the inner product $(\cdot, \cdot)$ defined in (8). □

Note that $(A')' = A$. Let S' be the modular matrix $(S_{\lambda',\mu})$ from A' to A.

LEMMA 5.3. *$S^t = S'$.*

Proof. When $A = X_N^{(1)}$ or $A = A_{2n}^{(2)}$, the lemma is clear since $J_\lambda(t_\mu) = J_\mu(t_\lambda)$.

We now assume $A = X_N^{(r)}$ with $r > 1$ and $A \neq A_{2n}^{(2)}$. Note that $M = \mathring{Q}$ (resp. $M' = \mathring{Q}'$). moreover

$$\check{h}' = \check{h}, \quad \epsilon(\mathring{P}') = \check{\mathring{P}}, \text{ and } \epsilon(\mathring{Q}') = \check{\mathring{Q}}.$$

It follows that ϵ induces the isomorphism

$$\mathring{P}'/(k+\check{h}')\mathring{Q}' \simeq \check{\mathring{P}}/(k+\check{h})\check{\mathring{Q}}.$$

From (7), we observe that $J_\lambda(t_{\mu'}) = J_{\mu'}(t_\lambda)$ for any $\lambda \in P_k$ and $\mu' \in P'_k$. It concludes the Lemma. $\square$

For any dominant integral weight Λ of $\mathfrak{g}(A)$ of level k, we denote by $\mathcal{H}_\Lambda$ the irreducible integrable representation of highest weight Λ. Let χ_Λ denote the normalized character of $\mathfrak{g}(A)$, i.e. $\chi_\Lambda = e^{-m_\Lambda \delta}\mathrm{ch}_{\mathcal{H}_\Lambda}$, where $\mathrm{ch}_{\mathcal{H}_\Lambda}$ is the character of $\mathcal{H}_\Lambda$ and the number m_Λ is the so-called modular anomaly (cf.[**Ka**, §12.7.5]). Let $V(A,k)$ be the vector space spanned by the normalized characters χ_Λ of level k. Then the matrix S gives a linear transformation from $V(A,k)$ to $V(A',k)$. For precise transformation formula, see [**Ka**, Theorem 13.8, 13.9]. In the literature the formula (20) is called Kac-Peterson formula (cf.[**Ka**, **Ka1**]) for the modular S-matrix as the linear transformation from $V(A,k)$ to $V(A',k)$.

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