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Conformal blocks, Verlinde formula and diagram automorphisms

Jiuzu Hong

Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA

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The Verlinde formula computes the dimension of the space of conformal blocks associated to simple Lie algebras and stable pointed curves. If a simply-laced simple Lie algebra admits a nontrivial diagram automorphism, then this automorphism acts on the space of conformal blocks naturally. We prove an analogue of the Verlinde formula for the trace of the diagram automorphism on the space of conformal blocks. Along the way, we get an analogue of the Kac-Walton formula for the trace of the diagram automorphism. We also get a twining type formula between the conformal blocks for the pair (sl_{2n+1}, sp_{2n}).

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E-mail address: jiuzu@email.unc.edu.

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1. Introduction

The Verlinde formula computes the dimension of the space of conformal blocks. It is fundamentally important in conformal field theory and algebraic geometry. The formula was originally conjectured by Verlinde [35] in conformal field theory. It was mathematically derived by combining the efforts of mathematicians including Tsuchiya-Ueno-Yamada [33], Faltings [4]. It was proved by Beauville-Laszlo [2], Kumar-Narasimhan-Ramanathan [23], Faltings [4], that conformal blocks can be identified with the generalized theta functions on the moduli stack of G-bundles on projective curves where G is a simply-connected simple algebraic group. Therefore the Verlinde formula also computes the dimension of the spaces of generalized theta functions. For a survey on Verlinde formula, see Sorger's Bourbaki talk [30].

Let $(C, \vec{p})$ be a stable k-pointed curve. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Let ℓ be a positive integer. Put

$$P_\ell = \{\lambda \in P^+ \mid \langle \lambda, \check{\theta} \rangle \leq \ell \rangle\}, \tag{1}$$

where θ is the highest root of $\mathfrak{g}$ and $\check{\theta}$ is the coroot of θ. Given a tuple of dominant weights $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ such that $\lambda_i \in P_\ell$ for each i. We can attach the space $V_{\mathfrak{g},\ell,\vec{\lambda}}(C, \vec{p})$ of conformal blocks of level ℓ to $(C, \vec{p})$ and $\vec{\lambda}$. We will recall the definition of conformal blocks in Section 3.4.

Let σ be a diagram automorphism on a simple Lie algebra $\mathfrak{g}$. One can attach another simple Lie algebra $\mathfrak{g}_\sigma$ as the orbit Lie algebra of $\mathfrak{g}$ (see Section 2 for details). If σ is trivial, then $\mathfrak{g} = \mathfrak{g}_\sigma$. Let Φ (resp. Φ_σ) be the set of roots of G (resp. G_σ). We put

$$\Delta = \prod_{\alpha \in \Phi} (e^\alpha - 1), \quad \Delta_\sigma = \prod_{\alpha \in \Phi_\sigma} (e^\alpha - 1). \tag{2}$$

There is a natural correspondence between σ-invariant weights (resp. dominant weights) of $\mathfrak{g}$ and weights (resp. dominant weights) of $\mathfrak{g}_\sigma$ (see Section 2.1). In this introduction we will identify them if no confusion occurs. For any dominant weight λ of $\mathfrak{g}$ (resp. $\mathfrak{g}_\sigma$), we denote by V_λ (resp. W_λ) the irreducible representation of $\mathfrak{g}$ (resp. $\mathfrak{g}_\sigma$) of highest weight λ. Let $\check{h}$ (resp. $\check{h}_\sigma$) be the dual Coxeter number of $\mathfrak{g}$ (resp. $\mathfrak{g}_\sigma$). Let G (resp. G_σ) be the associated simply-connected simple algebraic group of $\mathfrak{g}$ (resp. $\mathfrak{g}_\sigma$). Let T (resp. T_σ) be a maximal torus of G (resp. G_σ). Let W (resp. W_σ) denote the Weyl group of G (resp. G_σ).

Throughout this paper, we denote by $\mathrm{tr}(A|V)$ the trace of an operator A on a finite dimensional vector space V. The following is the celebrated Verlinde formula.

Theorem 1.1 *(Verlinde formula). Let $(C,\vec{p})$ be a stable k-pointed curve of genus g. Given any tuple $\vec{\lambda}=(\lambda_1,\lambda_2,\cdots,\lambda_k)$ of dominant weights of $\mathfrak{g}$ such that $\lambda_i\in P_\ell$ for each i, we have*

$$\dim V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p}) = |T_\ell|^{g-1}\sum_{t\in T_\ell^{\mathrm{reg}}/W}\frac{\mathrm{tr}(t|V_{\vec{\lambda}})}{\Delta(t)^{g-1}}, \tag{3}$$

where $V_{\vec{\lambda}}$ denotes the tensor product $V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_k}$ of representations of $\mathfrak{g}$ and

$$T_\ell=\{t\in T\,|\,e^\alpha(t)=1,\ \alpha\in(\ell+\check{h})Q_l\}$$

is a finite abelian subgroup in the maximal torus T, T_ℓ^{reg} denotes the set of regular elements in T_ℓ and T_ℓ^{reg}/W denotes the set of W-orbits in T_ℓ^{reg}. Here Q_l denotes the lattice spanned by long roots of $\mathfrak{g}$, and for any $\alpha\in Q_l$, e^α is the associated character of T.

From now on we always assume σ is nontrivial. When the tuple $\vec{\lambda}$ of dominant weights of $\mathfrak{g}$ is σ-invariant, one can define a natural operator on the space $V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p})$ of conformal blocks, which we still denote by σ, see Section 3.4. A natural question is how to compute the trace of σ as an operator on the space of the conformal blocks. In this paper, we derive a formula for the trace of σ, which is very similar to the Verlinde formula for the dimension of the space of conformal blocks. Very surprisingly, in the formula the role of $\mathfrak{g}$ is replaced by $\mathfrak{g}_\sigma$. The following is the main theorem of this paper.

Theorem 1.2. *Let $(C,\vec{p})$ be a stable k-pointed curve of genus g. Let σ be a nontrivial diagram automorphism on a simple Lie algebra $\mathfrak{g}$ which has dual Coxeter number $\check{h}$. Given a tuple $\vec{\lambda}=(\lambda_1,\lambda_2,\cdots,\lambda_k)$ of σ-invariant dominant weights of $\mathfrak{g}$ such that for each i, $\lambda_i\in P_\ell$, we have the following formula*

$$\mathrm{tr}(\sigma|V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p})) = |T_{\sigma,\ell}|^{g-1}\sum_{t\in T_{\sigma,\ell}^{\mathrm{reg}}/W_\sigma}\frac{\mathrm{tr}(t|W_{\vec{\lambda}})}{\Delta_\sigma(t)^{g-1}}, \tag{4}$$

where $W_{\vec{\lambda}}$ *denotes the tensor product* $W_{\vec{\lambda}} := W_{\lambda_1} \otimes \cdots \otimes W_{\lambda_k}$ *of representations of* $\mathfrak{g}_\sigma$ *and*

$$T_{\sigma,\ell} = \{t \in T_\sigma \mid e^\alpha(t) = 1, \alpha \in (\ell + \check{h})Q^\sigma\}.$$

Here $T^{\mathrm{reg}}_{\sigma,\ell}$ *denotes the set of regular elements in* $T_{\sigma,\ell}$*, and* $T^{\mathrm{reg}}_{\sigma,\ell}/W_\sigma$ *denotes the set of* W*-orbits in* $T^{\mathrm{reg}}_{\sigma,\ell}$ *and*

$$Q^\sigma = \begin{cases} \text{root lattice of } \mathfrak{g}_\sigma & \text{if } \mathfrak{g} \neq A_{2n} \\ \text{weight lattice of } \mathfrak{g}_\sigma & \text{if } \mathfrak{g} = A_{2n}. \end{cases}$$

Since the space of conformal blocks can be identified with the space of generalized theta functions, Theorem 1.2 implies the same formula for the trace of diagram automorphisms on the space of generalized theta functions.

Remark 1.3. By the basic representation theory of finite groups, we have the following formula

$$\dim V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p})^\sigma = \frac{1}{r}\sum_{i=1}^{r} \mathrm{tr}(\sigma^i | V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p})),$$

where r is the order of σ, and $V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p})^\sigma$ denotes the space of σ-invariants in $V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p})$. Combining Theorem 1.1 and Theorem 1.2, we immediately get a formula for the dimension of $V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p})^\sigma$.

The proof of Theorem 1.2 will be completed in Section 5.5. Our proof closely follows [4,1,22] for the derivation of the usual Verlinde formula, where the fusion ring plays essential role. In the standard approach to the Verlinde formula for general stable pointed curves, the factorization rules for conformal blocks and degeneration of projective smooth curves allow a reduction to projective line with three points case. Our basic idea is that we replace the dimension by the trace of the diagram automorphism everywhere. In our taking trace setting, we explain in Section 3.4 that factorization rules for conformal blocks and degeneration of curves are compatible well with the trace operation on the space of conformal blocks.

By replacing the dimension by the trace, we introduce σ-twisted fusion rings $R_\ell(\mathfrak{g},\sigma)$ in Section 3.5. We also introduce the σ-twisted representation ring $R(\mathfrak{g},\sigma)$ of $\mathfrak{g}$ (see Section 5.1). For the usual fusion ring $R_\ell(\mathfrak{g})$ and the representation ring $R(\mathfrak{g})$, it is important to establish a ring homomorphism from $R(\mathfrak{g})$ to $R_\ell(\mathfrak{g})$. Similarly, we establish a ring homomorphism from $R(\mathfrak{g},\sigma)$ to $R_\ell(\mathfrak{g},\sigma)$ in Section 5.3. One of important technical tools is that we interpret σ-twisted fusion product via affine analogue of Borel-Weil-Bott theorem, where the new product is introduced in Section 5.2. A vanishing theorem of Lie algebra cohomology by Teleman [32] plays a key role in our arguments as in the dimension setting (cf. [22, Chapter 4]). We describe all characters of the ring $R_\ell(\mathfrak{g}, \sigma)$ in Section 5.4. The Verlinde formula for the trace of diagram automorphism will be a consequence of the characterization of the ring $R_\ell(\mathfrak{g}, \sigma)$ and the determination of the Casimir element in $R_\ell(\mathfrak{g}, \sigma)$. As a byproduct we obtain an analogue of Kac-Walton formula (Theorem 5.11) in Section 5.2.

In the process of proving the coincidence of two products in the ring $R_\ell(\mathfrak{g}, \sigma)$ and establishing the ring homomorphism from $R(\mathfrak{g}, \sigma)$ to $R_\ell(\mathfrak{g}, \sigma)$, some interesting sign problems occur on the higher cohomology groups of vector bundles on affine Grassmannian and affine flag variety, also in affine BBG-resolution and affine Kostant homologies. The resolution of these sign problems is very crucial for the characterization of the ring $R_\ell(\mathfrak{g}, \sigma)$.

Let $\mathcal{L}_\ell(V_\lambda)$ be the vector bundle on the affine Grassmannian Gr_G of G associated to the level ℓ and the representation V_λ of G. By affine Borel-Weil-Bott theorem (cf. [21]) there is only one nonzero cohomology $H^i(\mathrm{Gr}_G, \mathcal{L}_\ell(V_\lambda))$ and the restricted dual $H^i(\mathrm{Gr}_G, \mathcal{L}_\ell(V_\lambda))^\vee$ is the irreducible integrable representation $\mathcal{H}_\lambda$ of the affine Lie algebra $\hat{\mathfrak{g}}$ of level ℓ. The action of σ on the highest weight vectors of $H^i(\mathrm{Gr}_G, \mathcal{L}_\ell(V_\lambda))^\vee$ is determined in Section 4.1 and Section 4.2. This problem is closely related to similar problem on the cohomology of line bundle on affine flag variety. The answer is very similar to the finite-dimensional situation which is due to Naito [28] where Lefschetz fixed point formula is used. In the affine setting, we don't know how to apply Lefschetz fixed point formula since the affine Grassmannian and affine flag variety are infinite-dimensional. Instead, our method is inspired by Lurie's short proof of Borel-Weil-Bott theorem [26]. Our method should be applicable to similar problems for general symmetrizable Kac-Moody groups. Similar sign problems also appear in BGG resolution and the Kostant homology for affine Lie algebras. See the discussions in Section 4.3.

Our starting point of this work is the Jantzen's twining formula (cf. [18,10,7,24,28,29]) relating representations of $\mathfrak{g}$ and $\mathfrak{g}_\sigma$, where the term "twining" is coined by Fuchs-Schellekens-Schweigert [7]. Given a σ-invariant dominant weight λ of $\mathfrak{g}$ where σ is the diagram automorphism as above. There is a unique operator σ on V_λ such that σ preserves the highest weight vector $v_\lambda \in V_\lambda$ and for any $u \in \mathfrak{g}$ and $v \in V_\lambda$, $\sigma(u \cdot v) = \sigma(u) \cdot \sigma(v)$. For any σ-invariant weight μ, Jantzen [18] proved the following formula

$$\mathrm{tr}(\sigma | V_\lambda(\mu)) = \dim W_\lambda(\mu),$$

where λ and μ are also regarded as (dominant) weights of $\mathfrak{g}_\sigma$. Given a tuple $\vec{\lambda}$ of σ-invariant dominant weights of $\mathfrak{g}$. Let $V_{\vec{\lambda}}^{\mathfrak{g}}$ (resp. $W_{\vec{\lambda}}^{\mathfrak{g}_\sigma}$) be the tensor invariant space of $\mathfrak{g}$ (resp. $\mathfrak{g}_\sigma$). Induced from the action of σ on each V_{λ_i}, σ acts on $V_{\vec{\lambda}}^{\mathfrak{g}}$ diagonally. Shen and the author obtained the following twining formula in the setting of tensor invariant spaces in [13],

$$\mathrm{tr}(\sigma | V_{\vec{\lambda}}^{\mathfrak{g}}) = \dim W_{\vec{\lambda}}^{\mathfrak{g}_\sigma}. \tag{5}$$

A consequence of (5) is that the σ-twisted representation ring $R(\mathfrak{g},\sigma)$ of $\mathfrak{g}$ is isomorphic to the representation ring $R(\mathfrak{g}_\sigma)$ of $\mathfrak{g}_\sigma$. This is how we are able to express the trace of σ on the space of conformal blocks by the data associated to $\mathfrak{g}_\sigma$.

It is well-known that given a tuple $\vec{\lambda}$ of dominant weights of $\mathfrak{g}$, the space $V_{\mathfrak{g},\ell,\vec{\lambda}}(\mathbb{P}^1,\vec{p})$ of conformal blocks on $(\mathbb{P}^1,\vec{p})$ stabilizes to the tensor co-invariant space $(V_{\vec{\lambda}})_{\mathfrak{g}}$ when the level ℓ increases. From Formula (5), it is natural to hope that the conformal blocks associated to $\mathfrak{g}$ and the conformal blocks associated to $\mathfrak{g}_\sigma$ are related and have a twining formula with a fixed level. Unfortunately this is not the case. We found the following counter-example using [31] (joint with P. Belkale).

Example 1.4. We have

$$\dim V_{sl_6,4,\lambda,\mu,\nu}(\mathbb{P}^1,0,1,\infty)=4,$$

where $\lambda=\omega_2+\omega_3+\omega_4$, $\mu=\omega_1+\omega_3+\omega_5$ and $\nu=\omega_1+2\omega_3+\omega_5$. Here $\omega_1,\omega_2,\omega_3,\omega_4,\omega_5$ denote the fundamental weights of sl_6. Since the order of σ on sl_6 is 2, this forces that the trace $\mathrm{tr}(\sigma|V_{sl_6,4,\lambda,\mu,\nu}(\mathbb{P}^1,0,1,\infty))$ is even. On the other hand, we have

$$\dim V_{so_7,4,\lambda,\mu,\nu}(\mathbb{P}^1,0,1,\infty)=3,$$

where $\lambda=\omega_{\sigma,2}+\omega_{\sigma,3}$, $\mu=\omega_{\sigma,1}+\omega_{\sigma,3}$ and $\nu=\omega_{\sigma,1}+2\omega_{\sigma,3}$. Here $\omega_{\sigma,1},\omega_{\sigma,2},\omega_{\sigma,3}$ denotes the fundamental weights of so_7.

Actually from formula (4) in Theorem 1.2, it is quite clear that $\mathrm{tr}(\sigma|V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p}))$ should not be the same as $\dim V_{\mathfrak{g}_\sigma,\ell,\vec{\lambda}}(C,\vec{p})$. Nevertheless, for the special pair (sl_{2n+1},sp_{2n}) we do have a twining formula where one needs to take different levels on both sides.

Theorem 1.5. *If ℓ is an odd positive integer, then the following formula holds*

$$\mathrm{tr}(\sigma|V_{sl_{2n+1},\ell,\vec{\lambda}}(C,\vec{p}))=\dim V_{sp_{2n},\frac{\ell-1}{2},\vec{\lambda}}(C,\vec{p}). \tag{6}$$

This theorem is a corollary of Theorem 1.2, and the proof will be given in Section 5.6. It has following interesting numerical consequences where ℓ is assumed to be odd.

- The trace $\mathrm{tr}(\sigma|V_{sl_{2n+1},\ell,\vec{\lambda}}(C,\vec{p}))$ is non-negative.
- If $\dim V_{sl_{2n+1},\ell,\vec{\lambda}}(C,\vec{p})$ is 1, then $\dim V_{sp_{2n},\frac{\ell-1}{2},\vec{\lambda}}(C,\vec{p})$ is also 1.
- If $V_{sp_{2n},\frac{\ell-1}{2},\vec{\lambda}}(C,\vec{p})$ is nonempty, then $V_{sl_{2n+1},\ell,\vec{\lambda}}(C,\vec{p})$ is nonempty.

Theorem 1.5 establishes a bridge between the conformal blocks for sl_{2n+1} and sp_{2n}.

The failure of the formula (6) in general is not really the end of the story. The combinatoric data appearing in the formula in Theorem 1.2 actually suggests a close connection with twisted affine Lie algebras. It is very natural from the point of view of the twining formula for affine Lie algebras by Fuchs-Schellekens-Schweigert [7]. Moreover the

σ-twisted fusion ring $R_\ell(\mathfrak{g}, \sigma)$ defined in this paper is closely related to Kac-Peterson formula for S-matrices of twisted affine Lie algebras (cf. [19]). The analogue of Kac-Walton formula obtained in this paper is also a strong hint. In fact this perspective has recently been clarified in [11] by the author. The connection on the trace of diagram automorphism on the space of conformal blocks and certain conformal field theory related to twisted affine Lie algebra was predicted by Fuchs-Schweigert [5]. It seems to the author that this work has confirmed Conjecture 2 in [6] when the automorphism is induced from a diagram automorphism of $\mathfrak{g}$. This work should also be closely related to the fusion rules for the orbifold conformal field theory that is developed by Birke-Fuchs-Schweigert [3] and Ishikawa-Tani [16].

A general theory of twisted conformal blocks has been developed recently by S. Kumar and the author [12]. It would be interesting to investigate the connection between this paper and [12].

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2. The root systems and affine Weyl group of orbit Lie algebras

2.1. Root systems

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Let I be the set of vertices of the Dynkin diagram of $\mathfrak{g}$. For each $i \in I$, let α_i (resp. ω_i) be the corresponding simple root (resp. fundamental weight). Let P be the weight lattice of $\mathfrak{g}$ and let P^+ be the set of dominant weights of $\mathfrak{g}$. Let Φ (resp. $\check{\Phi}$) be the set of roots (resp. coroots) of $\mathfrak{g}$, and let Q (resp. $\check{Q}$) be the root lattice (resp. coroot lattice) of $\mathfrak{g}$. For each root $\alpha \in \Phi$, let $\check{\alpha} \in \check{\Phi}$ be the associated coroot of α. Let $\langle , \rangle : P \times \check{Q} \to \mathbb{Z}$ be the perfect pairing between weight lattice and coroot lattices. Note that the matrix $(\langle \alpha_i, \check{\alpha}_j \rangle)$ is the Cartan matrix of $\mathfrak{g}$.

We denote by e_i, f_i, h_i the corresponding Chevalley generators in $\mathfrak{g}$ for each $i \in I$. Let σ be a nontrivial diagram automorphism of the Dynkin diagram of $\mathfrak{g}$. Note that $\mathfrak{g}$ can only be of types A_n, D_n, E_6 when σ is nontrivial. The automorphism σ acts on P, such that $\sigma(\alpha_i) = \alpha_{\sigma(i)}$ and $\sigma(\omega_i) = \omega_{\sigma(i)}$ for each $i \in I$. Clearly σ maps each dominant weight to another dominant weight.

The diagram automorphism σ defines an automorphism σ of the Lie algebra $\mathfrak{g}$ such that $\sigma(e_i) = e_{\sigma(i)}$, $\sigma(f_i) = f_{\sigma(i)}$, $\sigma(h_i) = h_{\sigma(i)}$ for each $i \in I$. Here we use the same notation σ to denote these automorphisms if no confusion occurs.

Let I_σ be the set of orbits of σ on I. There exists a unique simple Lie algebra $\mathfrak{g}_\sigma$ over $\mathbb{C}$ whose vertices of Dynkin diagram is indexed by I_σ (cf. [13, Section 2.2]), and the Cartan matrix is given as follows,

$$a_{\imath\jmath} = \begin{cases} \frac{|\imath|}{2} a_{ij}, & \mathfrak{g} \text{ is of type } A_{2n} \text{ and } \imath \text{ is disconnected} \\ |\imath| a_{ij}, & \text{otherwise} \end{cases}$$

for any $\imath \neq \jmath \in I_\sigma$, where $i \in \imath, j \in \jmath$ and $|\imath|$ is the cardinality of the $\imath$. The Lie algebra $\mathfrak{g}_\sigma$ is called the orbit Lie algebra of $(\mathfrak{g}, \sigma)$ in literature.

Let $\alpha_\imath$ (resp. $\check{\alpha}_\imath$) be the corresponding simple root (resp. simple coroot) for $\imath \in I_\sigma$. Let P_σ be the weight lattice of $\mathfrak{g}_\sigma$. There exists a bijection of lattices $\iota : P^\sigma \simeq P_\sigma$ such that $\iota^{-1}(\omega_\imath) = \sum_{i\in\imath} \omega_i$ for each $\imath \in I_\sigma$, where P^σ is the fixed point lattice of σ on P. Let ρ (resp. ρ_σ) be the sum of all fundamental weights of $\mathfrak{g}$ (resp. $\mathfrak{g}_\sigma$). Note that $\rho \in P^\sigma$, and $\iota(\rho) = \rho_\sigma$. Moreover,

$$\iota^{-1}(\alpha_\imath) = \begin{cases} \sum_{i\in\imath} \alpha_i & \text{for any } i \neq j \in \imath, a_{ij} = 0 \\ 2(\alpha_i + \alpha_j) & \imath = \{i, j\}, a_{ij} = -1 \end{cases} . \tag{7}$$

Let Q_σ (resp. $\check{Q}_\sigma$) be the root lattice (resp. coroot lattice) of $\mathfrak{g}_\sigma$. There is a projection map $\check{\iota} : \check{Q} \to \check{Q}_\sigma$. Under this projection, we have

$$\check{\iota}(\check{\alpha}_i) = \check{\alpha}_\imath, \qquad \text{for any } i \in \imath.$$

For any $\lambda \in P_\sigma$ and $x \in \check{Q}_\sigma$, we have the following compatibility

$$\langle \iota(\lambda), \check{\iota}(x) \rangle = \langle \lambda, x \rangle_\sigma, \tag{8}$$

where $\langle , \rangle_\sigma : P_\sigma \times \check{Q}_\sigma \to \mathbb{Z}$ is the perfect pairing between the weight lattice and dual root lattice for $\mathfrak{g}_\sigma$. The following is a table of $\mathfrak{g}$ and $\mathfrak{g}_\sigma$ for nontrivial σ ([13, Section 2.2] or [27, 6.4]):

(1) If $\mathfrak{g} = A_{2n-1}$ and σ is of order 2, then $\mathfrak{g}_\sigma = B_n$, $n \geq 2$.
(2) If $\mathfrak{g} = A_{2n}$ and σ is of order 2, then $\mathfrak{g}_\sigma = C_n$, $n \geq 1$, where C_1 by convention means A_1.
(3) If $\mathfrak{g} = D_n$ and σ is of order 2, then $\mathfrak{g}_\sigma = C_{n-1}$, $n \geq 4$.
(4) If $\mathfrak{g} = D_4$ and σ is of order 3, then $\mathfrak{g}_\sigma = G_2$.
(5) If $\mathfrak{g} = E_6$ and σ is of order 2, then $\mathfrak{g}_\sigma = F_4$.

Let θ be the highest root of $\mathfrak{g}$. It is clear that $\sigma(\theta) = \theta$.

Lemma 2.1. *We have*

$$\iota(\theta) = \begin{cases} \theta_{\sigma,s} & (\mathfrak{g}, \mathfrak{g}_\sigma) \neq (A_{2n}, C_n) \\ \frac{1}{2}\theta_\sigma & (\mathfrak{g}, \mathfrak{g}_\sigma) = (A_{2n}, C_n) \end{cases} \tag{9}$$

where θ_σ is the highest root of $\mathfrak{g}_\sigma$ and $\theta_{\sigma,s}$ is the highest short root of $\mathfrak{g}_\sigma$. Moreover,

$$\check{\iota}(\check{\theta}) = \begin{cases} \check{\theta}_\sigma & (\mathfrak{g}, \mathfrak{g}_\sigma) \neq (A_{2n}, C_n) \\ 2\check{\theta}_{\sigma,s} & (\mathfrak{g}, \mathfrak{g}_\sigma) = (A_{2n}, C_n) \end{cases} \tag{10}$$

where $\check{\theta}_\sigma$ (resp. $\check{\theta}_{\sigma,s}$) is the highest coroot (the coroot of the highest root) of $\mathfrak{g}_\sigma$.

Proof. We first determine $\check{\iota}(\check{\theta})$. Let $\check{\mathfrak{g}}$ be the Lie algebra with root system dual to the root system of $\mathfrak{g}$. We still denote by σ the diagram automorphism on $\check{\mathfrak{g}}$ induced from the diagram automorphism σ on $\mathfrak{g}$. It is well-known that the root system of $\mathfrak{g}_\sigma$ is dual to the root system of the fixed Lie algebra $\check{\mathfrak{g}}^\sigma$ (cf. [10,13]).

By [10, Lemma 4.3], σ acts on the highest root subspace $\check{\mathfrak{g}}_{\check{\theta}}$ by 1 if $\mathfrak{g}$ is not of type A_{2n}; otherwise, σ acts on $\check{\mathfrak{g}}_{\check{\theta}}$ by -1. It follows that if $\mathfrak{g}$ is not A_{2n}, then $\check{\mathfrak{g}}_{\check{\theta}}$ is the highest root subspace of the fixed point Lie subalgebra $\check{\mathfrak{g}}^\sigma$. Thus, in this case $\check{\iota}(\check{\theta}) = \check{\theta}_\sigma$. When $\mathfrak{g}$ is of type A_{2n}, by [19, Prop. 8.3] $\check{\iota}(\check{\theta}) = 2\check{\theta}_{\sigma,s}$.

Finally, we can determine $\iota(\theta)$ from (7) and [14, Table 2, p. 88], and we get the formula (9). □

Note that $\check{\theta}_\sigma$ is the coroot of $\theta_{\sigma,s}$ and $\check{\theta}_{\sigma,s}$ is the coroot of θ_σ.

Lemma 2.2. *Let I_k be the Dynkin diagram of type C_k with $k \geq 2$, where I_k consists of vertices $i_1, i_2, \cdots, i_k$ such that the simple root α_{i_1} is a long root. Then the long root lattice Q_l of C_k is spanned by $\alpha_{i_1}, 2\alpha_{i_2}, \cdots, 2\alpha_{i_k}$.*

Proof. For any $k \geq 1$, let I_k be the Dynkin diagram of C_k (where $C_1 = A_1$), there exists a natural embedding $I_k \hookrightarrow I_{k+1}$. Assume that I_k consists of vertices $i_1, i_2, \cdots, i_k$, where the simple root α_{i_1} is the long root. Let θ_k be the highest long root of C_k. Then $\theta_{k+1} - \theta_k = 2\alpha_{k+1}$. Therefore the lattice of long roots of C_k for $k \geq 2$, is spanned by $\alpha_{i_1}, 2\alpha_{i_2}, \cdots, 2\alpha_{i_k}$. □

Let Q^σ denote the lattice of σ-invariant elements in the root lattice Q of $\mathfrak{g}$.

Lemma 2.3. *With respect to the isomorphism $\iota : P_\sigma \simeq P^\sigma$, we have*

$$\iota(Q^\sigma) = \begin{cases} Q_\sigma & \text{if } \mathfrak{g} \text{ is not of type } A_{2n} \\ P_\sigma = \frac{1}{2}Q_{\sigma,l} & \text{if } \mathfrak{g} \text{ is } A_{2n} \end{cases}$$

where $Q_{\sigma,l}$ is the lattice spanned by the long roots of $\mathfrak{g}_\sigma$.

Proof. Clearly Q^σ has a basis $\{\sum_{i\in\imath}\alpha_i \,|\, \imath\in I_\sigma\}$. In view of (7), it is easy to see when $\mathfrak{g}$ is not of type A_{2n}, $\iota(Q^\sigma)$ is the root lattice Q_σ of $\mathfrak{g}_\sigma$.

If $\mathfrak{g}$ is of type A_2, this is just a direct simple calculation. Otherwise, if $\mathfrak{g}=A_{2n}$ with $n\geq 2$, then $\iota(Q^\sigma)=\sum_{\imath\in I_\sigma} a_\imath\mathbb{Z}\alpha_\imath$, where

$$a_\imath=\begin{cases}1 & \text{if } \imath \text{ is not connected}\\ \frac{1}{2} & \text{if } \imath \text{ is connected}\end{cases}.$$

Let $\imath_0$ be the connected σ-orbit in I. Note that $\imath_0$ corresponds to the long root of C_n. By Lemma 2.2, our lemma follows. $\square$

2.2. Affine Weyl groups and diagram automorphisms

In this subsection, we refer to [15] the basics of affine Weyl groups and alcoves.

Let W be the Weyl group of $\mathfrak{g}$. The group W acts on the weight lattice P. Let $P_\mathbb{R}$ be the space $P\otimes_\mathbb{Z}\mathbb{R}$. For each root $\alpha\in\Phi$, let s_α be the corresponding reflection in W, i.e. for any $\lambda\in P_\mathbb{R}$, $s_\alpha(\lambda)=\lambda-\langle\lambda,\check{\alpha}\rangle\alpha$.

Let W_ℓ be the affine Weyl group $W\ltimes \ell Q$ for any $\ell\in\mathbb{Q}$. Since $\mathfrak{g}$ is simply-laced, the Coxeter number is equal to the dual Coxeter number, moreover all roots have the same length. For any $\ell\in\mathbb{N}$, W_ℓ is the Weyl group of the affine Lie algebra $\hat{\mathfrak{g}}$ of level ℓ. Let s_0 be the affine reflection $s_{\theta,1}$, i.e.

$$s_{\theta,1}(\lambda)=\lambda-(\langle\lambda,\check{\theta}\rangle-\ell)\theta, \tag{11}$$

where θ is the highest root of $\mathfrak{g}$. The affine Weyl group W_ℓ is a Coxeter group generated by $\{s_i \mid i\in\hat{I}\}$. For any $\alpha\in\Phi$ and $n\in\ell\mathbb{Z}$, the hyperplane

$$H_{\alpha,n}=\{\lambda\in P_\mathbb{R}\,|\,\langle\lambda,\check{\alpha}\rangle=n\}$$

is an affine wall of W_ℓ. Every component of the complements of affine walls in $P_\mathbb{R}$ is an alcove. The affine Weyl group W_ℓ acts on the set of alcoves simply and transitively. Let A_0 be the fundamental alcove, and it can be described as follows,

$$\{\lambda\in P_\mathbb{R}\,|\,\langle\lambda,\check{\alpha}_i\rangle>0,\ \text{ for any } i\in I,\ \text{ and } \langle\lambda,\check{\theta}\rangle<\ell\}.$$

The diagram automorphism σ acts on W. Let W^σ be the fixed point group of σ on W. Let W_σ be the Weyl group of $\mathfrak{g}_\sigma$ with simple reflections $\{s_\imath\,|\,\imath\in I_\sigma\}$. Then there exists an isomorphism $\iota: W^\sigma\simeq W_\sigma$ such that for any $\imath\in I_\sigma$,

$$\iota^{-1}(s_\imath)=\begin{cases}\prod_{i\in\imath}s_i & \text{any } i\neq j\in\imath,\ a_{ij}=0\\ s_is_js_i & \text{if } \imath=\{i,j\} \text{ and } a_{ij}=-1\end{cases}. \tag{12}$$

The following lemma is obvious.

Lemma 2.4. *The group action of W_σ on P_σ is compatible with the action of W^σ on P^σ, with respect to the isomorphisms $\iota : P^\sigma \simeq P_\sigma$ and $\iota : W^\sigma \simeq W_\sigma$.*

The diagram automorphism σ also acts naturally on W_ℓ. Let W_ℓ^σ denote the fixed point group of σ on W_ℓ. It is easy to see that

$$W_\ell^\sigma = W^\sigma \ltimes \ell Q^\sigma. \tag{13}$$

Let $\hat{I}_\sigma$ be the set $I_\sigma \sqcup \{0\}$. We have the following lemma (cf. [7, Section 5.2]).

Lemma 2.5. *W_ℓ^σ is a Coxeter group generated by $\{\iota^{-1}(s_\imath) \mid \imath \in \hat{I}_\sigma\}$.*

The group W_ℓ^σ naturally acts on $P_\mathbb{R}^\sigma$. Let $\mathcal{A}$ denote the set of alcoves of W_ℓ in $P_\mathbb{R}$. There exists a natural action of σ on $\mathcal{A}$. Let $\mathcal{A}^\sigma$ be the set of σ-stable alcoves.

Lemma 2.6.

(1) *For any $A \in \mathcal{A}^\sigma$, the set A^σ is not empty, where A^σ is the set of σ-invariant elements in A.*
(2) *For any two σ-stable alcoves A and A' in $\mathcal{A}$, there exists a unique $w \in W_\ell^\sigma$ such that $w(A) = A'$.*

Proof. We first prove (1). For any $\lambda \in A$, $\lambda, \sigma(\lambda), \cdots \sigma^{r-1}(\lambda) \in A$, where r is the order of σ. By the convexity of A,

$$\frac{\lambda + \sigma(\lambda) + \cdots + \sigma^{r-1}(\lambda)}{r} \in A,$$

which is σ-invariant.

Now we prove (2). The affine Weyl group W_ℓ acts simply and transitively on $\mathcal{A}$ (cf. [15, §4.5]). Hence, given any two elements $A, A' \in \mathcal{A}^\sigma$, there exists a unique $w \in W_\ell$ such that $w(A) = A'$. In particular, we have

$$\sigma(w)(A) = \sigma w \sigma^{-1}(A) = \sigma w(A) = \sigma(A') = A' = w(A).$$

By the uniqueness of w, we have $\sigma(w) = w$. □

Let $P_{\sigma,\mathbb{R}}$ be the space $P_\sigma \otimes_\mathbb{Z} \mathbb{R}$. We still denote by $\iota : W_\ell^\sigma \simeq W_\sigma \ltimes \iota(\ell Q^\sigma)$ the natural isomorphism of groups. By Lemma 2.3, $W_\sigma \ltimes \iota(\ell Q^\sigma)$ is an affine Weyl group. In view of Lemma 2.1 and Lemma 2.3,

$$A_{0,\sigma} = \{\lambda \in P_{\sigma,\mathbb{R}} \mid \langle \lambda, \check{\alpha}_\imath \rangle_\sigma > 0 \text{ for any } \imath \in I_\sigma, \text{ and } \langle \lambda, \check{\iota}(\check{\theta}) \rangle_\sigma < \ell\}$$

is the fundamental alcove of $W_\sigma \ltimes \iota(\ell Q^\sigma)$.

Let $\mathcal{A}_\sigma$ be the set of alcoves of $W_\sigma \ltimes \iota(\ell Q^\sigma)$ in $P_{\sigma,\mathbb{R}}$.

Proposition 2.7.

(1) *The isomorphism* $\iota : P^{\sigma}_{\mathbb{R}} \simeq P_{\sigma,\mathbb{R}}$ *induces a bijection* $\iota : (A_0)^{\sigma} \simeq A_{0,\sigma}$.
(2) *There exists a bijection* $\mathcal{A}^{\sigma} \simeq \mathcal{A}_{\sigma}$ *with the map given by*

$$A \mapsto \iota(A^{\sigma}).$$

(3) *For any* $\lambda \in P^{\sigma}_{\mathbb{R}}$, λ *is in an affine wall of* W_{ℓ} *if and only if* $\iota(\lambda) \in P_{\sigma,\mathbb{R}}$ *is in an affine wall of* $W_{\sigma} \ltimes \iota(\ell Q^{\sigma})$.

Proof. We first prove (1). For any $\lambda \in P^{\sigma}_{\mathbb{R}}$, $\lambda \in (A_0)^{\sigma}$ if and only if $\iota(\lambda) \in (A_0)^{\sigma}$, since

$$\langle \lambda, \check{\alpha}_i \rangle = \langle \iota(\lambda), \check{\iota}(\check{\alpha}_i) \rangle_{\sigma} = \langle \iota(\lambda), \check{\alpha}_{\imath} \rangle_{\sigma} > 0,$$

for any $\imath \in I_{\sigma}$ and $i \in \imath$, and $\langle \lambda, \check{\theta} \rangle = \langle \iota(\lambda), \check{\iota}(\check{\theta}) \rangle_{\sigma} < \ell$.

The second part (2) of proposition follows from Lemma 2.6 and the first part of the proposition. The third part (3) of the proposition follows from the first and second part of proposition. □

Let $\ell_{\sigma} : W^{\sigma}_{\ell} \to \mathbb{N}$ denote the length function on the Coxeter group W^{σ}_{ℓ}. For any $\lambda \in \ell Q^{\sigma}$, let τ_{λ} be the translation on $P^{\sigma}_{\mathbb{R}}$ by λ. The following lemma will be used in the proofs of Proposition 5.15 and Lemma 5.18 in Section 5.

Lemma 2.8. *The length* $\ell_{\sigma}(\tau_{\lambda})$ *is even.*

Proof. When $\mathfrak{g}$ is not A_{2n}, by Lemma 2.3 $\iota(Q^{\sigma}) = Q_{\sigma}$. The Coxeter group W^{σ}_{ℓ} is isomorphic to the affine Weyl group $W_{\sigma} \ltimes Q_{\sigma}$. The problem is reduced to show that for any $\lambda \in Q_{\sigma}$, τ_{λ} has even length in $W_{\sigma} \ltimes Q_{\sigma}$. If λ is dominant, then $\ell_{\sigma}(\tau_{\lambda}) = \langle \lambda, 2\check{\rho}_{\sigma} \rangle$ (cf. [17]), where $\check{\rho}_{\sigma}$ is the sum of all fundamental weights of $\mathfrak{g}_{\sigma}$. Hence $\ell_{\sigma}(\tau_{\lambda})$ is even. For general λ, $\lambda = w(\lambda)$ for some $w \in W_{\sigma}$ and some dominant weight $\lambda^{+} \in Q_{\sigma}$. Then $\tau_{\lambda} = w\tau_{\lambda^{+}}w^{-1}$, and hence τ_{λ} is even.

When $\mathfrak{g}$ is of type A_{2n}, by Lemma 2.3 $\iota(Q^{\sigma}) = \frac{1}{2}Q_{\sigma,l}$. The normalized Killing form on $\mathfrak{g}_{\sigma}$ can identify $\frac{\ell}{2}Q_{\sigma,l}$ with $\frac{\ell}{2}\check{Q}_{\sigma}$ (cf. [1, Proof of Lemma 9.3 (b)]), where $\check{Q}_{\sigma}$ is the coroot lattice of $\mathfrak{g}_{\sigma}$. This identification is compatible with the action of W_{σ}. Hence W^{σ}_{ℓ} is isomorphic to $W_{\sigma} \ltimes \check{Q}_{\sigma}$. By the same argument as above, the length $\ell_{\sigma}(\tau_{\lambda})$ is also even. □

3. Conformal blocks

3.1. Affine Lie algebra

Let $\mathfrak{g}$ be a simple Lie algebra. Let $\mathbb{C}((t))$ be the field of Laurent series over $\mathbb{C}$. Let $\tilde{\mathfrak{g}}$ be the associated affine Kac-Moddy algebra $\mathfrak{g}((t)) \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $\mathfrak{g}((t))$ denotes the loop Lie algebra $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((t))$. The Lie bracket $[,]$ on $\tilde{\mathfrak{g}}$ is given by

$$[u \otimes f, v \otimes g] := [u, v] \otimes fg + (u|v)\mathrm{Res}_{t=0}(\frac{df}{dt}g)c,$$

and $[u \otimes t^n, d] = nu \otimes t^n$, $\quad [d, c] = 0$, $\quad [u \otimes f, c] = 0$, for any $u, v \in \mathfrak{g}$ and $f, g \in \mathbb{C}((t))$, where $[u, v]$ is the Lie bracket on $\mathfrak{g}$ and $(\cdot|\cdot)$ is the normalized invariant bilinear form on $\mathfrak{g}$. For convenience, we identify $u \otimes 1$ with u for any $u \in \mathfrak{g}$, and hence $\mathfrak{g}$ is naturally a Lie subalgebra of $\tilde{\mathfrak{g}}$.

Put $\hat{\mathfrak{g}} = \mathfrak{g}((t)) \oplus \mathbb{C}c$. Clearly $\hat{\mathfrak{g}}$ is a Lie subalgebra of $\tilde{\mathfrak{g}}$. The affine Kac-Moody algebra $\tilde{\mathfrak{g}}$ corresponds to the extended Dynkin diagram $\hat{I} = I \sqcup \{0\}$ of $\mathfrak{g}$. The Cartan subalgebra $\tilde{\mathfrak{t}}$ associated to $\tilde{\mathfrak{g}}$ is $\mathfrak{t} \oplus \mathbb{C}c \oplus \mathbb{C}d$. For any $\lambda \in P$ we view it as a weight of $\tilde{\mathfrak{g}}$ in the following way, λ extends to $\tilde{\mathfrak{t}}$ such that $\lambda(d) = \lambda(c) = 0$. Let δ be the linear functional on $\tilde{\mathfrak{t}}$ such that

$$\delta|_{\mathfrak{t}} = 0, \quad \delta(c) = 0, \quad \delta(d) = 1.$$

Let $\alpha_0 = -\theta + \delta$, where θ is the highest root of $\mathfrak{g}$. Then $\{\alpha_i \,|\, i \in \hat{I}\}$ is the set of simple roots of $\tilde{\mathfrak{g}}$. The fundamental weight Λ_0 of $\tilde{\mathfrak{g}}$ is given by the linear functional on $\tilde{\mathfrak{t}}$ such that

$$\Lambda_0|_{\mathfrak{t}} = 0, \quad \Lambda_0(c) = 1, \quad \Lambda_0(d) = 0.$$

3.2. Affine Weyl groups and Weyl groups of affine Kac-Moody algebras

In the following we describe the relationship between the affine Weyl groups of simple Lie algebras and the Weyl groups of affine Kac-Moody algebras. For more details, one can refer to [19, §6]. These two different perspectives are both crucial in this paper.

Let $\hat{W}$ be the Weyl group of the affine Kac-Moody algebra $\tilde{\mathfrak{g}}$ (cf. [19, §3.7]). Set $\tilde{\mathfrak{t}}^*_{\mathbb{R}} := P_{\mathbb{R}} + \mathbb{R}\Lambda_0 + \mathbb{R}\delta$. The Weyl group $\hat{W}$ acts on $\tilde{\mathfrak{t}}^*_{\mathbb{R}}$. Note that $\hat{W}$ keep δ invariant (cf. [19, §6.5]). Hence the Weyl group $\hat{W}$ acts on $\hat{P}_{\mathbb{R},\ell}$ for any $\ell \in \mathbb{R}$, where

$$\hat{P}_{\mathbb{R},\ell} := \{x \in \tilde{\mathfrak{t}}^*_{\mathbb{R}} \,|\, \langle x, c\rangle = \ell\}/\mathbb{R}\delta.$$

With respect to the isomorphism $P_{\mathbb{R}} \simeq \hat{P}_{\mathbb{R},\ell}$ given by $\lambda \mapsto \lambda + \ell\Lambda_0$, we have the following lemma (cf. [19, §6.5,§6.6]).

Lemma 3.1. *There exists an isomorphism* af $: \hat{W} \simeq W_\ell$ *of groups such that for any* $\Lambda = \lambda + \ell\Lambda_0 \in \hat{P}_{\mathbb{R},\ell}$ *and* $w \in \hat{W}$, *the following formula holds,*

$$w \cdot \Lambda = \mathrm{af}(w) \cdot \lambda + \ell\Lambda_0 \ \textit{in}\ \hat{P}_{\mathbb{R},\ell}.$$

Let $\hat{\rho}$ be the sum $\sum_{i\in\hat{I}} \Lambda_i$ of all fundamental weights of $\tilde{\mathfrak{g}}$. By [19, §6.2.8], $\hat{\rho} = \rho + \check{h}\Lambda_0$ where ρ is the sum $\sum_{i\in I} \omega_i$ of all fundamental weights of $\mathfrak{g}$, and $\check{h}$ is the dual Coxeter number of $\mathfrak{g}$.

We define $\star$ action of $\hat{W}$ on $\hat{P}_{\mathbb{R},\ell}$ as follows,

$$w \star \Lambda = w \cdot (\Lambda + \hat{\rho}) - \hat{\rho}, \quad w \in \hat{W}, \Lambda \in \hat{P}_{\mathbb{R},\ell}.$$

Similarly, we still denote by $\star$ the following action of W_ℓ on $P_{\mathbb{R}}$,

$$w \star \lambda = w \cdot (\lambda + \rho) - \rho, \quad w \in W_\ell, \lambda \in P_{\mathbb{R}}.$$

Lemma 3.2. *Given $\Lambda = \lambda + \ell\Lambda_0 \in \hat{P}_{\mathbb{R},\ell}$ and $w \in \hat{W}$, we have*

$$w \star \Lambda = \mathrm{af}(w) \star \lambda + \ell\Lambda_0, \qquad \textit{where } \mathrm{af}(w) \in W_{\ell+\check{h}}.$$

Proof. It follows from Lemma 3.1 and the formula $\hat{\rho} = \rho + \check{h}\Lambda_0$. $\square$

3.3. Diagram automorphisms as intertwining operators of representations

We denote by V_λ the irreducible representation of $\mathfrak{g}$ of highest weight λ for each $\lambda \in P^+$. From now on we always fix a highest weight vector $v_\lambda \in V_\lambda$ for each λ. There exists a unique operator $\sigma : V_\lambda \to V_{\sigma(\lambda)}$ such that $\sigma(v_\lambda) = v_{\sigma(\lambda)}$, and $\sigma(u \cdot v) = \sigma(u) \cdot \sigma(v)$ for any $u \in \mathfrak{g}$ and $v \in V_\lambda$.

When $\sigma(\lambda) = \lambda$, σ acts on V_λ. Given any σ-invariant dominant weight of $\mathfrak{g}$ and any r-th root of unity $\xi \in \mathbb{C}$ where r is the order of σ, we denote by $V_{\lambda,\xi}$ the representation of $\mathfrak{g} \rtimes \langle\sigma\rangle$, i.e. it consists of V_λ as representation of $\mathfrak{g}$ and an operator $\sigma : V_\lambda \to V_\lambda$ such that σ acts on v_λ by ξ, and $\sigma(u \cdot v) = \sigma(u) \cdot \sigma(v)$.

Given a tuple $\vec{\lambda} = (\lambda_1, \cdots, \lambda_k)$ of dominant weights of $\mathfrak{g}$. We denote by $V_{\vec{\lambda}}$ the tensor product $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_k}$. Denote by $V_{\vec{\lambda}}^{\mathfrak{g}}$ the invariant space of $\mathfrak{g}$ on $V_{\vec{\lambda}}$. The collection of operators $\{\sigma : V_\lambda \to V_{\sigma(\lambda)}\}$ induce

$$\sigma : V_{\vec{\lambda}} \to V_{\sigma(\vec{\lambda})}, \quad \sigma : V_{\vec{\lambda}}^{\mathfrak{g}} \to V_{\sigma(\vec{\lambda})}^{\mathfrak{g}},$$

where $\sigma(\vec{\lambda}) = (\sigma(\lambda_1), \cdots, \sigma(\lambda_k))$.

Recall the set P_ℓ defined in (1). The following lemma is well-known (cf. [19, §12.4]).

Lemma 3.3. *For any $\lambda \in P^+$ and $\ell \in \mathbb{N}$, $\lambda + \ell\Lambda_0$ is a dominant weight of $\tilde{\mathfrak{g}}$ if and only if $\lambda \in P_\ell$.*

For any $\lambda \in P_\ell$, let $\hat{M}(V_\lambda)$ denote the generalized Verma module $U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{p}})} V_\lambda$ of $\hat{\mathfrak{g}}$, where $\hat{\mathfrak{p}} = \mathfrak{g}[[t]] \oplus \mathbb{C} \cdot c$ acts on V_λ by evaluating $t = 0$ and c acts by ℓ. Then the unique maximal irreducible quotient $\mathcal{H}_\lambda$ is an irreducible integrable representation of $\hat{\mathfrak{g}}$ of level ℓ. The action of $\hat{\mathfrak{g}}$ on $\mathcal{H}_\lambda$ extends uniquely to the irreducible integrable representation of $\tilde{\mathfrak{g}}$ of highest weight $\lambda + \ell\Lambda_0$ by letting d act trivially on the highest weight vectors.

From the construction of $\mathcal{H}_\lambda$, there exists a natural inclusion $V_\lambda \to \mathcal{H}_\lambda$. Denote by $\bar{v}_\lambda$ the image of $v_\lambda \in V_\lambda$ in $\mathcal{H}_\lambda$, which is again a highest weight vector in $\mathcal{H}_\lambda$.

The diagram automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ extends to an automorphism on $\hat{\mathfrak{g}}$ (by abuse of notation we still use σ) such that $\sigma(u \otimes f) = \sigma(u) \otimes f$ for any $u \in \mathfrak{g}$ and $f \in \mathbb{C}((t))$, and $\sigma(c) = c$. As in the case of V_λ, there exists a unique operator $\sigma : \mathcal{H}_\lambda \to \mathcal{H}_{\sigma(\lambda)}$ such that $\sigma(\bar{v}_\lambda) = \bar{v}_{\sigma(\lambda)}$, and $\sigma(X \cdot v) = \sigma(X)\sigma(v)$ for any $X \in \hat{\mathfrak{g}}$ and $v \in \mathcal{H}_\lambda$. In particular σ acts on $\mathcal{H}_\lambda$ when $\sigma(\lambda) = \lambda$. As in the case of V_λ, for any σ-invariant dominant weight λ of $\mathfrak{g}$ and for any r-th root of unity ξ, we denote by $\mathcal{H}_{\lambda,\xi}$ the representation of $\hat{\mathfrak{g}} \rtimes \langle \sigma \rangle$ which satisfies similar conditions for $V_{\lambda,\xi}$.

Given a tuple $\vec{\lambda}$ of dominant weights, denote by $\mathcal{H}_{\vec{\lambda}}$ the tensor product $\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_k}$. The operators $\{\sigma : \mathcal{H}_\lambda \to \mathcal{H}_{\sigma(\lambda)}\}$ induce the operator $\sigma : \mathcal{H}_{\vec{\lambda}} \to \mathcal{H}_{\sigma(\vec{\lambda})}$ such that

$$\sigma(v_1 \otimes \cdots \otimes v_k) = \sigma(v_1) \otimes \cdots \otimes \sigma(v_k),$$

for any $v_i \in \mathcal{H}_{\lambda_k}$, $i = 1, \cdots, k$.

The inclusion $V_\lambda \hookrightarrow \mathcal{H}_\lambda$ is compatible with the diagram automorphism, i.e.

$$\begin{tikzcd} V_\lambda \arrow[r] \arrow[d, "\sigma"] & \mathcal{H}_\lambda \arrow[d, "\sigma"] \\ V_{\sigma(\lambda)} \arrow[r] & \mathcal{H}_{\sigma(\lambda)} \end{tikzcd} \quad . \tag{14}$$

Let $\hat{\mathfrak{g}}^-$ denote the Lie subalgebra $t^{-1}\mathfrak{g}[t^{-1}]$. We denote by $(\mathcal{H}_\lambda)_{\hat{\mathfrak{g}}^-}$ the coinvariant space of $\mathcal{H}_\lambda$ with respect to the action of $\hat{\mathfrak{g}}^-$. The Lie algebra $\mathfrak{g}$ acts naturally on $(\mathcal{H}_\lambda)_{\hat{\mathfrak{g}}^-}$. The following lemma is well-known.

Lemma 3.4. *As representations of $\mathfrak{g}$, we have a natural isomorphism $V_\lambda \simeq (\mathcal{H}_\lambda)_{\hat{\mathfrak{g}}^-}$. Moreover the following diagram commutes*

$$\begin{tikzcd} V_\lambda \arrow[r] \arrow[d, "\sigma"] & (\mathcal{H}_\lambda)_{\hat{\mathfrak{g}}^-} \arrow[d, "\sigma"] \\ V_{\sigma(\lambda)} \arrow[r] & (\mathcal{H}_{\sigma(\lambda)})_{\hat{\mathfrak{g}}^-} \end{tikzcd} \quad . \tag{15}$$

Let τ be the Cartan involution of $\mathfrak{g}$ such that $\tau(e_i) = -f_i, \tau(f_i) = -e_i, \tau(h_i) = -h_i$, where e_i, f_i, h_i for $i \in I$, are Chevalley generators of $\mathfrak{g}$. Then τ is an automorphism on $\mathfrak{g}$. For any finite dimensional representation V of $\mathfrak{g}$, by composing τ we can redefine a new representation structure on V, $X * v := \tau(X) \cdot v$, for any $X \in \mathfrak{g}$ and $v \in V$. We denote by V^τ this τ-twisted representation.

For any dominant weight λ, let λ^* be the dominant weight $-\omega_0(\lambda)$ where ω_0 is the longest element in the Weyl group W. The space V_λ^τ is isomorphic to V_{λ^*} as representations of $\mathfrak{g}$.

The Cartan involution τ on $\mathfrak{g}$ extends to an automorphism on $\hat{\mathfrak{g}}$ (by abuse of notation we still use τ) such that $\tau(u \otimes f) = \tau(u) \otimes f$ and $\tau(c) = c$ for any $u \in \mathfrak{g}, f \in \mathbb{C}((t))$.

Denote by $\mathcal{H}^\tau_\lambda$ the representation of $\hat{\mathfrak{g}}$ by composing the automorphism $\tau : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$. Then $\mathcal{H}^\tau_\lambda \simeq \mathcal{H}_{\lambda^*}$.

Summarizing the above discussions, we have the following lemma.

Lemma 3.5.

(1) *There exists a unique* $\mathbb{C}$*-linear isomorphism* $\tau_\lambda : V_\lambda \to V_{\lambda^*}$ *such that*

$$\tau_\lambda(v_\lambda) = v_{\lambda^*}, \quad \tau_\lambda(u \cdot v) = \tau(u) \cdot \tau_\lambda(v), \quad \text{for any } u \in \mathfrak{g} \text{ and } v \in V_\lambda.$$

(2) *There exists a unique* $\mathbb{C}$*-linear isomorphism* $\tau_\lambda : \mathcal{H}_\lambda \to \mathcal{H}_{\lambda^*}$ *such that*

$$\tau_\lambda(v_\lambda) = v_{\lambda^*}, \quad \tau_\lambda(X \cdot v) = \tau(X) \cdot \tau_\lambda(v), \quad \text{for any } X \in \hat{\mathfrak{g}} \text{ and } v \in \mathcal{H}_\lambda.$$

The isomorphism $\tau_\lambda : V_\lambda \to V_{\lambda^*}$ for each λ induces an isomorphism $\tau_{\vec{\lambda}} : V^{\mathfrak{g}}_{\vec{\lambda}} \to V^{\mathfrak{g}}_{\vec{\lambda}^*}$ for any tuple of dominant weights $\vec{\lambda}$. Since for any weight λ, we have $\sigma(\lambda^*) = \sigma(\lambda)^*$, and $\sigma \circ \tau = \tau \circ \sigma$, we have the following lemma.

Lemma 3.6. *Let* $\vec{\lambda}^*$ *denote* $(\lambda^*_1, \cdots, \lambda^*_k)$*. The following diagram commutes:*

$$\begin{tikzcd} V^{\mathfrak{g}}_{\vec{\lambda}} \arrow[r, "\tau"] \arrow[d, "\sigma"] & V^{\mathfrak{g}}_{\vec{\lambda}^*} \arrow[d, "\sigma"] \\ V^{\mathfrak{g}}_{\sigma(\vec{\lambda})} \arrow[r, "\tau"] & V^{\mathfrak{g}}_{\sigma(\vec{\lambda})^*} \end{tikzcd},$$

3.4. Conformal blocks and diagram automorphisms

A k-pointed projective curve consists of a projective curve C and k-distinct smooth points $\vec{p} = (p_1, \cdots, p_k)$ in C. Given a k-pointed projective curve $(C, \vec{p})$, we associate a dominant weight $\lambda_i \in P_\ell$ to each point p_i. Let $\mathfrak{g}(C \backslash \vec{p})$ be the space of $\mathfrak{g}$-valued regular functions on $C \backslash \vec{p}$. The space $\mathfrak{g}(C \backslash \vec{p})$ is naturally a Lie algebra induced from $\mathfrak{g}$. There exists a Lie algebra action of $\mathfrak{g}(C \backslash \vec{p})$ on $\mathcal{H}_{\vec{\lambda}}$, and the space $V_{\mathfrak{g},\ell,\vec{\lambda}}(C, \vec{p})$ of conformal blocks associated to $\vec{p}$ and $\vec{\lambda}$ is defined as follows:

$$V_{\mathfrak{g},\ell,\vec{\lambda}}(C, \vec{p}) := (\mathcal{H}_{\vec{\lambda}})_{\mathfrak{g}(C \backslash \vec{p})} = \mathcal{H}_{\vec{\lambda}} / \mathfrak{g}(C \backslash \vec{p}) \mathcal{H}_{\vec{\lambda}}.$$

Let $\tau_{\vec{\lambda}} : \mathcal{H}_{\vec{\lambda}} \to \mathcal{H}_{\vec{\lambda}^*}$ be the $\mathbb{C}$-linear isomorphism $\tau_{\lambda_1} \otimes \cdots \otimes \tau_{\lambda_k}$. The map $\tau_{\vec{\lambda}}$ descends to an isomorphism on the space of conformal blocks

$$\tau_{\vec{\lambda}} : V_{\mathfrak{g},\ell,\vec{\lambda}}(C, \vec{p}) \to V_{\mathfrak{g},\ell,\vec{\lambda}^*}(C, \vec{p}).$$

Lemma 3.7. *We have the following commutative diagram:*

$$\begin{tikzcd} V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p}) \arrow[r, "\tau_{\vec{\lambda}}"] \arrow[d, "\sigma"] & V_{\mathfrak{g},\ell,\vec{\lambda}^*}(C,\vec{p}) \arrow[d, "\sigma"] \\ V_{\mathfrak{g},\ell,\sigma(\vec{\lambda})}(C,\vec{p}) \arrow[r, "\tau_{\vec{\lambda}}"] & V_{\mathfrak{g},\ell,\sigma(\vec{\lambda}^*)}(C,\vec{p}) \end{tikzcd} .$$

Proof. The automorphism σ commutes with the automorphism τ on $\mathfrak{g}$, i.e. $\tau\circ\sigma = \sigma\circ\tau$. Then commutativity easily follows. $\square$

Proposition 3.8. *Let $\vec{p} = \{p_1, p_2, \cdots, p_s\}$, $\vec{q} = \{q_1, q_2, \cdots, q_t\}$ be two finite nonempty subsets smooth points of C, without common points; let $\lambda_1, \cdots, \lambda_s, \mu_1, \cdots, \mu_t$ be elements in P_ℓ. We let $\mathfrak{g}(C\backslash\vec{p})$ act on V_{μ_j} through the evaluation map $X\otimes f \mapsto f(q_j)X$. The inclusions $V_{\mu_j} \hookrightarrow \mathcal{H}_{\mu_j}$ induce an isomorphism*

$$(\mathcal{H}_{\vec{\lambda}} \otimes V_{\vec{\mu}})_{\mathfrak{g}(C\backslash\vec{p})} \simeq (\mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_{\vec{\mu}})_{\mathfrak{g}(C\backslash\vec{p}\cup\vec{q})} = V_{\mathfrak{g},\ell,\vec{\lambda},\vec{\mu}}(C,\vec{p},\vec{q}), \tag{16}$$

and this isomorphism is compatible with the diagram automorphism σ, i.e. the following diagram commutes

$$\begin{tikzcd} (\mathcal{H}_{\vec{\lambda}} \otimes V_{\vec{\mu}})_{\mathfrak{g}(C\backslash\vec{p})} \arrow[r, "\simeq"] \arrow[d, "\sigma"] & V_{\mathfrak{g},\ell,\vec{\lambda},\vec{\mu}}(C,\vec{p},\vec{q}) \arrow[d, "\sigma"] \\ (\mathcal{H}_{\sigma(\vec{\lambda})} \otimes V_{\sigma(\vec{\mu})})_{\mathfrak{g}(C\backslash\vec{p})} \arrow[r, "\simeq"] & V_{\mathfrak{g},\ell,\sigma(\vec{\lambda}),\sigma(\vec{\mu})}(C,\vec{p},\vec{q}) \end{tikzcd} . \tag{17}$$

Proof. Isomorphism (16) is a well-known theorem (cf. [1, Proposition 2.3]). The commutativity of diagram (17) follows from the commutativity (15). $\square$

When $\vec{q} = q$ and $\mu = 0$. Isomorphism (16) is the so-called "propogation of vacua". Proposition 3.8 shows that the propagation of vacua is compatible with the action of the diagram automorphism.

Lemma 3.9.

(1) *For any $p \in \mathbb{P}^1$, one has $V_{\mathfrak{g},\ell}(\mathbb{P}^1) \simeq V_{\mathfrak{g},\ell,0}(\mathbb{P}^1, p) \simeq \mathbb{C}$ by 1, and the automorphism σ acts on $V_{\mathfrak{g},\ell}(\mathbb{P}^1)$ and $V_{\mathfrak{g},\ell,0}(\mathbb{P}^1, p)$ by 1.*
(2) *For any $p \neq q$ in $\mathbb{P}^1$, one has $V_{\mathfrak{g},\ell,\lambda,\lambda^*}(\mathbb{P}^1, p, q) \simeq (V_\lambda \otimes V_{\lambda^*})_{\mathfrak{g}} = \mathbb{C}$. As a consequence, the automorphism σ acts on $V_{\mathfrak{g},\ell,\lambda,\lambda^*}(\mathbb{P}^1, p, q)$ by 1 for any σ-invariant dominant weight λ.*

Proof. By Proposition 3.8, there exists a map $\mathbb{C} \to V_{\mathbb{P}^1}(p;0)$ compatible with the action of σ where $\mathbb{C}$ is viewed as a trivial representation of $\mathfrak{g}$ and σ acts on $\mathbb{C}$ trivially. By [1, Corollary 4.4], this map is an isomorphism. By Proposition 3.8 again, $V_{\mathfrak{g},\ell}(\mathbb{P}^1) \simeq V_{\mathfrak{g},\ell,0}(\mathbb{P}^1,p) \simeq \mathbb{C}$ and this isomorphism is also compatible with the action of σ. Hence, σ acts on $V_{\mathfrak{g},\ell}(\mathbb{P}^1)$ and $V_{\mathfrak{g},\ell,0}(\mathbb{P}^1,p)$ by 1. This proves (1).

Similarly, by Proposition 3.8 there exists a map $(V_\lambda \otimes V_{\lambda^*})_\mathfrak{g} \to V_{\mathfrak{g},\ell,\lambda,\lambda^*}(\mathbb{P}^1,p,q)$ which is compatible with the action of σ. This map is an isomorphism in view of [1, Corollary 4.4]. On the other hand, it is easy to see that σ acts on $(V_\lambda \otimes V_{\lambda^*})_\mathfrak{g}$ by 1. Hence it also acts on $V_{\mathfrak{g},\ell,\lambda,\lambda^*}(\mathbb{P}^1,p,q)$ by 1. $\square$

Given a stable k-pointed curve $(C,\vec{p})$. Assume that $q \in C$ is a nodal point in C. Let $\pi : \tilde{C} \to C$ be the normalization of C at q. Denote by $\{q_+, q_-\}$ the preimage of q via π. Without confusion, we will still denote by $p_1, \cdots, p_k$ the preimages of $p_1, p_2, \cdots, p_k \in C$ in $\tilde{C}$.

We choose a system of $\mathfrak{g}$-equivariant maps $\kappa_\mu : \mathbb{C} \to V_\mu \otimes V_{\mu^*}$ for $\mu \in P^+$, such that the following diagram commutes

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\kappa_\mu"] \arrow[dr, "\kappa_{\sigma(\mu)}"'] & V_\mu \otimes V_{\mu^*} \arrow[d, "\sigma"] \\ & V_{\sigma(\mu)} \otimes V_{\sigma(\mu^*)} \end{tikzcd} ,$$

for any dominant weight μ. Note that the map κ_μ induces the following map

$$\hat{\kappa}_\mu : V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p}) \simeq V_{\mathfrak{g},\ell,\vec{\lambda},0}(C,\vec{p},q) \to V_{\mathfrak{g},\ell,\vec{\lambda},\mu,\mu^*}(C,\vec{p},q_+,q_-).$$

Moreover, it is easy to see that the following diagram commutes

$$\begin{tikzcd} V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p}) \arrow[r, "\hat{\kappa}_\mu"] \arrow[d, "\sigma"] & V_{\mathfrak{g},\ell,\vec{\lambda},\mu,\mu^*}(C,\vec{p},q_+,q_-) \arrow[d, "\sigma"] \\ V_{\mathfrak{g},\ell,\sigma(\vec{\lambda})}(C,\vec{p}) \arrow[r, "\hat{\kappa}_{\sigma(\mu)}"] & V_{\mathfrak{g},\ell,\sigma(\vec{\lambda}),\sigma(\mu),\sigma(\mu^*)}(C,\vec{p},q_+,q_-) \end{tikzcd} . \tag{18}$$

Theorem 3.10. *The map*

$$V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p}) \xrightarrow{(\hat{\kappa}_\mu)} \bigoplus_{\mu \in P_\ell} V_{\mathfrak{g},\ell,\vec{\lambda},\mu,\mu^*}(\tilde{C},\vec{p},q_+,q_-) \tag{19}$$

is an isomorphism. Moreover the following diagram commutes,

$$\begin{tikzcd} V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p}) \arrow[r, "(\hat{\kappa}_\mu)"] \arrow[d, "\sigma"] & \bigoplus_{\mu\in P_\ell} V_{\mathfrak{g},\ell,\vec{\lambda},\mu,\mu^*}(\tilde{C},\vec{p},q_+,q_-) \arrow[d, "\sigma"] \\ V_{\mathfrak{g},\ell,\sigma(\vec{\lambda})}(C,\vec{p}) \arrow[r, "(\hat{\kappa}_\mu)"] & \bigoplus_{\mu\in P_\ell} V_{\mathfrak{g},\ell,\sigma(\vec{\lambda}),\sigma(\mu),\sigma(\mu^*)}(\tilde{C},\vec{p},q_+,q_-) \end{tikzcd} . \tag{20}$$

Proof. Isomorphism (19) is the well-known factorization theorem on conformal blocks (cf. [34, Theorem 3.19]), and the commutativity (20) follows from the commutativity (18). □

Recall that P_ℓ^σ is the set of σ-invariant dominant weights in P_ℓ.

Corollary 3.11. *With the same setup as above. If $\sigma(\vec{\lambda})=\vec{\lambda}$, then the following equality holds*

$$\operatorname{tr}(\sigma|V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p}))=\sum_{\mu\in P_\ell^\sigma}\operatorname{tr}(\sigma|V_{\mathfrak{g},\ell,\vec{\lambda},\mu,\mu^*}(\tilde{C},\vec{p},q_+,q_-)).$$

Proof. This is an immediate consequence of Theorem 3.10. □

Given a family $(\pi:\mathcal{C}\to X,\vec{\mathfrak{p}})$ of stable k-pointed curves where π is a family of projective curves with at most nodal singularities over a smooth variety X and $\vec{\mathfrak{p}}=(\mathfrak{p}_1,\cdots,\mathfrak{p}_k)$ is a collection of sections $\mathfrak{p}_i:X\to\mathcal{C}$ with disjoint images such that $\mathfrak{p}_i(x)$ is a smooth point in $C_x:=\pi^{-1}(x)$ for each i and $x\in X$, one can attach a sheaf of conformal blocks $\mathcal{V}_{\mathfrak{g},\ell,\vec{\lambda}}(\mathcal{C},\vec{\mathfrak{p}})$ on X which is locally free and of finite rank, see [25] for the coordinate-free approach to the sheaf of conformal blocks. For each $x\in X$, the fiber $\mathcal{V}_{\mathfrak{g},\ell,\vec{\lambda}}(\mathcal{C},\vec{\mathfrak{p}})|_x$ is the space of conformal blocks $V_{\mathfrak{g},\ell,\vec{\lambda}}(C_x,\vec{\mathfrak{p}}(x))$, where $\vec{\mathfrak{p}}(x)=(\mathfrak{p}_1(x),\cdots,\mathfrak{p}_k(x))$ are the k-distinct smooth points in C_x as the image of x via $\vec{\mathfrak{p}}$.

From the construction the sheaf of conformal blocks (cf. [25]), one can see the diagram automorphism σ acts algebraically on $\mathcal{V}_{\mathfrak{g},\ell,\vec{\lambda}}(\mathcal{C},\vec{\mathfrak{p}})$. Denote by $\langle\sigma\rangle$ the cyclic group generated by σ. Then the group $\langle\sigma\rangle$ is isomorphic to $\mathbb{Z}/r\mathbb{Z}$, where r is the order of σ.

Lemma 3.12. *For any family $(\pi:\mathcal{C}\to X,\vec{\mathfrak{p}})$ of stable pointed curves, the function $x\in X\mapsto\operatorname{tr}(\sigma|V_{\mathfrak{g},\ell,\vec{\lambda}}(C_x,\vec{\mathfrak{p}}(x)))$ is constant.*

Proof. Given any irreducible representation ρ of $\langle\sigma\rangle$, we denote by $\operatorname{ch}(\rho)$ and $\operatorname{ch}(V_{\mathfrak{g},\ell,\vec{\lambda}}(C_x,\vec{\mathfrak{p}}(x)))$ the characters of ρ and $V_{\mathfrak{g},\ell,\vec{\lambda}}(C_x,\vec{\mathfrak{p}}(x))$ respectively as representations of $\langle\sigma\rangle$. For any two functions ϕ,ψ on $\langle\sigma\rangle$, we define the bilinear form

$$(\phi,\psi)=\frac{1}{r}\sum_{i=0}^{r-1}\phi(\sigma^i)\psi(\sigma^{-i}),$$

where r is the order of σ. For any $x\in X$, let $m_\rho(x)$ be the multiplicity of ρ appearing in $V_{\mathfrak{g},\ell,\vec{\lambda}}(C_x,\vec{\mathfrak{p}}(x))$. By representation theory of finite groups, we have

$$m_\rho(x) = (\mathrm{ch}\rho, \mathrm{ch}V_{\mathfrak{g},\ell,\vec{\lambda}}(C_x, \vec{\mathfrak{p}}(x))).$$

This is a continuous function on X with integer values. This is forced to be constant. Hence

$$\mathrm{tr}(\sigma|V_{\mathfrak{g},\ell,\vec{\lambda}}(C_x, \vec{\mathfrak{p}}(x))) = \sum m_\rho(x)\mathrm{tr}(\sigma|\rho)$$

is constant along $x \in X$. $\square$

The following theorem shows that the trace of the diagram automorphism on the space of conformal blocks satisfies factorization properties.

Theorem 3.13.

(1) *For any stable k-pointed curve $(C, \vec{p})$, let $\vec{\lambda}$ be a tuple of dominant weights in P_ℓ^σ attached to $\vec{p}$. Then the value $\mathrm{tr}(\sigma|V_{\mathfrak{g},\ell,\vec{\lambda}}(C, \vec{p}))$ only depends on $\vec{\lambda}$ and the genus of C.*

(2) *Given a stable k-pointed curve $(C, \vec{p})$ of genus $g \geq 1$ and a stable $(k+2)$-pointed curve $(C', \vec{q})$ of genus $g-1$. We have the following formula*

$$\mathrm{tr}(\sigma|V_{\mathfrak{g},\ell,\vec{\lambda}}(C, \vec{p})) = \sum_{\mu \in P_\ell^\sigma} \mathrm{tr}(\sigma|V_{\mathfrak{g},\ell,\vec{\lambda},\mu,\mu^*}(C', \vec{q})),$$

where a tuple $\vec{\lambda} = (\lambda_1, \cdots, \lambda_k)$ of dominant weights in P_ℓ^σ, is attached to $\vec{p}$ and the first k points of $\vec{q}$.

(3) *Given any tuples of dominant weights $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_s)$ and $\vec{\mu} = (\mu_1, \cdots, \mu_t)$ in P_ℓ^σ where $s, t \geq 2$, we have the following equality*

$$\mathrm{tr}(\sigma|V_{\mathfrak{g},\ell,\vec{\lambda},\vec{\mu}}(\mathbb{P}^1, \vec{p}_1)) = \bigoplus_{\nu \in P_\ell^\sigma} \mathrm{tr}(\sigma|V_{\mathfrak{g},\ell,\vec{\lambda},\nu}(\mathbb{P}^1, \vec{p}_2))\mathrm{tr}(\sigma|V_{\mathfrak{g},\ell,\vec{\mu},\nu^*}(\mathbb{P}^1, \vec{p}_3)),$$

where $\vec{p}_1$ is any tuple of $s+t$ distinct points, $\vec{p}_2$ is any tuple of $s+1$ distinct points and $\vec{p}_3$ is any tuple of $t+1$ distinct points in $\mathbb{P}^1$.

Proof. We first prove part (1). By the standard theory of moduli of curves (cf. [9, Theorem 2.15]), there exists a chain of families of stable k-pointed curves over smooth bases connecting any two stable k-pointed curves with the same genus. In view of Lemma 3.12, (1) follows.

From the theory of moduli of curves again (cf. [9, Theorem 2.15]) and the dimension formula for the space of nodal curves with fixed nodal types (see the discussions after [9, Theorem 2.15]), we know that any smooth pointed stable curve can be degenerated to an irreducible stable pointed curve with only one nodal point. Then part (2) follows from part (1) and Corollary 3.11.

We now proceed to prove part (3). Let C be the union of two projective lines $C = C_1 \cup C_2$ where C_1 and C_2 intersect at one point z. Let $\vec{p} = (p_1, \cdots, p_s)$ be a set of s distinct points in $C_1 \backslash \{z\}$ and $\vec{q} = \{q_1, \cdots, q_t\}$ be another set of t distinct points in $C_2 \backslash \{z\}$ where $s, t \geq 2$. Clearly $(C, \vec{p} \cup \vec{q})$ is a stable $(s+t)$-pointed curve of genus zero. Again by the theory of moduli of curves, there exists a family $\pi : \mathcal{C} \to X$ of stable $(s+t)$-pointed curves over a smooth variety X such that $C_{x_0} = C$ with $\vec{p} \cup \vec{q}$ and any other fiber is a projective line with a tuple $\vec{p}_1$ of $s+t$ points. By Lemma 3.12,

$$\operatorname{tr}(\sigma | V_{\mathfrak{g},\ell,\vec{\lambda},\vec{\mu}}(C, \vec{p}, \vec{q})) = \operatorname{tr}(\sigma | V_{\mathfrak{g},\ell,\vec{\lambda},\vec{\mu}}(\mathbb{P}^1, \vec{p}_1).$$

Let $\pi : \tilde{C} \to C$ be the normalization of C at z with the preimage (z_+, z_-) of z. The pointed curve $(\tilde{C}, \vec{p}, \vec{q}, z_+, z_-) = (\mathbb{P}^1, \vec{p}, z_+) \sqcup (\mathbb{P}^1, \vec{q}, z_-)$ is a disjoint union of a $(s+1)$-points projective line and a $(t+1)$-pointed projective line. Finally, part (3) follows from Corollary 3.11 and Lemma 3.12. $\square$

Remark 3.14. By Theorem 3.13, the computation of the trace of the diagram automorphism on the space of conformal blocks can be reduced to the trace of the diagram automorphism on the space of conformal blocks on the pointed curve $(\mathbb{P}^1, (0, 1, \infty))$.

3.5. σ-twisted fusion ring

Let J be a finite set with an involution $\lambda \mapsto \lambda^*$. We denote by $\mathbb{N}^J$ the free commutative monoid generated by J, that is, the set of sums $\sum_{\lambda \in J} n_\lambda \lambda$ with $n_\lambda \in \mathbb{N}$. The involution of J extends by linearity to an involution $x \mapsto x^*$ of $\mathbb{N}^J$. We first recall the definition of fusion rule (cf. [1, §5]).

Definition 3.15. A fusion rule on J is a map $N : \mathbb{N}^J \to \mathbb{Z}$ satisfying the following conditions:

(1) One has $N(0) = 1$, and $N(\lambda) > 0$ for some $\lambda \in J$;
(2) $N(x^*) = N(x)$ for every $x \in \mathbb{N}^J$;
(3) For $x, y \in \mathbb{N}^J$, one has $N(x + y) = \sum_\lambda N(x + \lambda) N(y + \lambda^*)$.

The kernel of a fusion rule N by definition is the set of elements $\lambda \in J$ such that $N(\lambda + x) = 0$ for all $x \in \mathbb{N}^I$. A fusion rule on J is called non-degenerate if the kernel is empty.

Lemma 3.16. *If $\sigma(\vec{\lambda}) = \vec{\lambda}$, then the trace $\operatorname{tr}(\sigma | V_{\mathfrak{g},\ell,\vec{\lambda}}(C, \vec{p}))$ is an integer.*

Proof. When the order of σ is 2, this is obvious. In general, it follows from Theorem 5.11, Formula (5) in the introduction and part (3) of Theorem 3.13. $\square$

Theorem 3.17. *The map* $\mathrm{tr}_\sigma : \mathbb{N}^{P_\ell^\sigma} \to \mathbb{Z}$ *given by*

$$\sum \lambda_i \mapsto \mathrm{tr}(\sigma | V_{\mathfrak{g},\ell,\vec{\lambda}}(\mathbb{P}^1, \vec{p})),$$

where $\vec{\lambda} = (\lambda_1, \cdots, \lambda_k)$ *and* $\vec{p} = (p_1, \cdots, p_k)$ *is the set of any* k*-distinct points in* $\mathbb{P}^1$*, is a non-degenerate fusion rule. Here the set* P_ℓ^σ *is equipped with the involution* $\lambda \mapsto \lambda^* := -w_0(\lambda)$*, where* w_0 *is the longest element in the Weyl group* W*.*

Proof. By Lemma 3.16, the trace map tr_σ indeed always takes integer values.

Condition (1) of Definition 3.15 follows from part (1) of Lemma 3.9. Condition (2) follows from Lemma 3.7. Condition (3) follows from part (3) of Theorem 3.13. The non-degeneracy follows from part (2) of Lemma 3.9. □

Let $R_\ell(\mathfrak{g}, \sigma)$ be a free abelian group with the set P_ℓ^σ as a basis. As a consequence of Theorem 3.17 and [1, Proposition 5.3], we can define a ring structure on $R_\ell(\mathfrak{g}, \sigma)$ by putting

$$\lambda \cdot \mu := \sum_{\nu \in P_\ell^\sigma} \mathrm{tr}(\sigma | V_{\mathfrak{g},\ell,\lambda,\mu,\nu^*}(\mathbb{P}^1, 0, 1, \infty))\nu, \quad \text{for any } \lambda, \mu \in P_\ell^\sigma. \tag{21}$$

Let S_σ be the set of characters (i.e. ring homomorphisms) of $R_\ell(\mathfrak{g}, \sigma)$ into $\mathbb{C}$. The following proposition is a consequence of general facts on fusion ring by Beauville [1, Corollary 6.2].

Proposition 3.18.

(1) $R_\ell(\mathfrak{g}, \sigma) \otimes \mathbb{C}$ *is a reduced commutative ring.*
(2) *The map* $R_\ell(G, \sigma) \otimes \mathbb{C} \to \mathbb{C}^{S_\sigma}$ *given by* $x \mapsto (\chi(x))_{x \in S_\sigma}$ *is an isomorphism of* $\mathbb{C}$*-algebras.*
(3) *We have* $\chi(x^*) = \overline{\chi(x)}$*, where* $\overline{\chi(x)}$ *denotes the complex conjugation of* $\chi(x)$ *for any* $\chi \in S_\sigma$ *and* $x \in R_\ell(\mathfrak{g}, \sigma)$*.*

Let ω_σ be the Casimir element in $R_\ell(\mathfrak{g}, \sigma)$ defined as follows

$$\omega_\sigma = \sum_{\lambda \in P_\ell^\sigma} \lambda \cdot \lambda^*. \tag{22}$$

Proposition 3.19. *For any* k*-pointed stable curve* $(C, \vec{p})$ *and for any* σ*-invariant tuple* $\vec{\lambda}$ *of dominant weights in* P_ℓ*, we have the following formula*

$$\mathrm{tr}(\sigma | V_{\mathfrak{g},\ell,\vec{\lambda}}(C, \vec{p})) = \sum_{\chi \in S_\sigma} \chi(\lambda_1) \cdots \chi(\lambda_k) \chi(\omega_\sigma)^{g-1},$$

where g *is the genus of* C *and* $\chi(\omega_\sigma) = \sum_{\lambda \in P_\ell^\sigma} |\chi(\lambda)|^2$*.*

Proof. This is a consequence of part (2) of Theorem 3.13 and [1, Proposition 6.3]. □

From this proposition, if we can determine the set S_σ and the value $\chi(\omega_\sigma)$ for each $\chi \in S_\sigma$, then the trace $\mathrm{tr}(\sigma|V_{\mathfrak{g},\ell,\vec{\lambda}}(C,\vec{p}))$ is known.

4. Sign problems

4.1. Borel-Weil-Bott theorem on the affine flag variety

Let G be a connected and simply-connected simple algebraic group associated to a simple Lie algebra $\mathfrak{g}$. Let $G((t))$ be the loop group of G, and let $\hat{G}$ be the nontrivial central extension of $G((t))$ by the center $\mathbb{C}^\times$. Then $\hat{\mathfrak{g}}$ is the Lie algebra of $\hat{G}$. Let $\tilde{G}$ be the group $\tilde{G} = \hat{G} \rtimes \mathbb{C}^\times$ whose Lie algebra is the affine Kac-Moody algebra $\tilde{\mathfrak{g}}$.

Let $\mathcal{I}$ be the Iwahori subgroup of $G((t))$, i.e. $\mathcal{I} = \mathrm{ev}_0^{-1}(B)$, where B is the Borel subgroup of G. Let Fl_G be the affine flag variety $G((t))/\mathcal{I}$ of G. Let $\hat{\mathcal{I}}$ be the group $\mathcal{I} \times \mathbb{C}^\times$, where $\mathbb{C}^\times$ is the center of $\hat{G}$. Let $\tilde{\mathcal{I}}$ be the product $\hat{\mathcal{I}} \rtimes \mathbb{C}^\times$ as subgroup of $\tilde{G}$. Then we have

$$\mathrm{Fl}_G \simeq \hat{G}/\hat{\mathcal{I}} \simeq \tilde{G}/\tilde{\mathcal{I}}.$$

Given any algebraic representation V of $\tilde{\mathcal{I}}$, we can attach a $\tilde{G}$-equivariant vector bundle $\mathcal{L}(V)$ on Fl_G as $\mathcal{L}(V) := \tilde{G} \times_{\tilde{\mathcal{I}}} V^*$, where V^* is the dual representation of $\tilde{\mathcal{I}}$. Let Λ be a character of $\tilde{\mathcal{I}}$ and let $\mathbb{C}_\Lambda$ be the associated 1-dimensional representation of $\tilde{\mathcal{I}}$. We denote by $\mathcal{L}(\Lambda)$ the $\tilde{G}$-equivariant line bundle $\mathcal{L}(\mathbb{C}_\Lambda)$ on Fl_G.

For any ind-scheme X and any vector bundle $\mathcal{F}$ on X, the cohomology groups $H^*(X,\mathcal{F})$ carry a topology. We put $H^*(X,\mathcal{F})^\vee$ the restricted dual of $H^*(X,\mathcal{F})$, i.e. $H^*(X,\mathcal{F})^\vee$ consists of continuous functional on $H^*(X,\mathcal{F})$ where we take discrete topology on $\mathbb{C}$. The affine flag variety Fl_G is an ind-scheme of ind-finite type. We refer the reader to [21] for the foundation of flag varieties of Kac-Moody groups.

Recall the following affine analogue of Borel-Weil-Bott theorem (cf. [21, Theorem 8.3.11]).

Theorem 4.1. *Given any dominant weight Λ of $\tilde{G}$ and any $w \in \hat{W}$, the space $H^{\ell(w)}(\mathrm{Fl}_G, \mathcal{L}(w\star\Lambda))^\vee$ is naturally the integrable irreducible representation $\mathcal{H}_\Lambda$ of $\tilde{\mathfrak{g}}$ of highest weight Λ, where $w\star\Lambda = w\cdot(\Lambda+\hat{\rho})-\hat{\rho}$ and $H^{\ell(w)}(\mathrm{Fl}_G,\mathcal{L}(w\star\Lambda))$ is the cohomology of the line bundle $\mathcal{L}(w\star\Lambda)$ on Fl_G. Moreover, $H^i(\mathrm{Fl}_G,\mathcal{L}(w\star\Lambda)) = 0$ if $i \neq \ell(w)$.*

Let σ be a diagram automorphism on G. It induces an action on $G((t))$ by acting trivially on t. It also induces actions on $\hat{G}$ and $\tilde{G}$ by acting trivially on the center and degree component. Note that σ preserves $\tilde{\mathcal{I}}$. For any σ-invariant character Λ of $\tilde{\mathcal{I}}$, we have a natural σ-equivariant structure on $\mathcal{L}(\Lambda)$, since

$$\tilde{G} \rtimes \langle\sigma\rangle \times_{\tilde{\mathcal{I}}\rtimes\langle\sigma\rangle} (\mathbb{C}_\Lambda)^* \simeq \tilde{G} \times_{\tilde{\mathcal{I}}} (\mathbb{C}_\Lambda)^*,$$

where the action of σ on $\mathbb{C}_\Lambda$ is by the scalar 1. Let ξ be an r-th root of unity, where r is the order of σ. We denote by $\mathcal{L}(\Lambda,\xi)$ the following $\tilde{G} \rtimes \langle\sigma\rangle$-equivariant line bundle,

$$\mathcal{L}(\Lambda,\xi) := \tilde{G} \rtimes \langle\sigma\rangle \times_{\tilde{\mathcal{I}}\rtimes\langle\sigma\rangle} (\mathbb{C}_{\Lambda,\xi})^*$$

where $\hat{\mathcal{I}}$ acts on $\mathbb{C}_{\Lambda,\xi}$ by Λ and σ acts on $\mathbb{C}_{\Lambda,\xi}$ by ξ. By this convention the natural $\tilde{G} \rtimes \langle\sigma\rangle$-equivariant structure on $\mathcal{L}(\Lambda)$ is isomorphic to $\mathcal{L}(\Lambda,1)$.

For any σ-orbit $\imath$ in the affine Dynkin diagram $\hat{I}$, let $G_\imath$ be the simply-connected algebraic group associated to the sub-diagram $\imath$ and let $B_\imath$ be the Borel subgroup of $G_\imath$. We have the following possibilities

$$G_\imath = \begin{cases} \mathrm{SL}_2 & \imath = \{i\} \\ \mathrm{SL}_2 \times \mathrm{SL}_2 & \imath = \{i,j\} \text{ and } i,j \text{ are not connected} \\ \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 & \imath = \{i,j,k\} \text{ and } i,j,k \text{ are not connected} \\ \mathrm{SL}_3 & \imath = \{i,j\} \text{ and } i,j \text{ are connected} \end{cases}.$$

We still denote by σ the diagram automorphism on $G_\imath$ which preserves $B_\imath$. Any σ-invariant weight λ of $G_\imath$ can be written as $n\rho_\imath$ for some integer $n \in \mathbb{Z}$, where $\rho_\imath$ is the sum of all fundamental weights of $G_\imath$. Let $\mathcal{B}_\imath := G_\imath/B_\imath$ be the flag variety of $G_\imath$. Put $d_\imath = \dim G_\imath/B_\imath$.

As in the affine case for any r-th root of unity and any σ-invariant character λ of $B_\imath$, we set

$$\mathcal{L}(\lambda,\xi) = G_\imath \times_{B_\imath} (\mathbb{C}_{\lambda,\xi})^*$$

as a $G_\imath \rtimes \langle\sigma\rangle$-equivariant line bundle on $\mathcal{B}_\imath$. Let $\Omega_\imath$ be the canonical bundle of $\mathcal{B}_\imath$. Note that the canonical bundle $\Omega_\imath$ is naturally a $G_\imath \rtimes \langle\sigma\rangle$-equivariant line bundle.

Lemma 4.2. *We have the following isomorphism of $G_\imath \rtimes \langle\sigma\rangle$-equivariant line bundles $\Omega_\imath \simeq \mathcal{L}(-2\rho_\imath, \epsilon_\imath)$, where $\epsilon_\imath = (-1)^{d_\imath - 1}$.*

Proof. The canonical bundle $\Omega_\imath$ is naturally isomorphic to $G_\imath \times_{B_\imath} (\wedge^{d_\imath}(\mathfrak{g}_\imath/\mathfrak{b}_\imath))^*$, where $\mathfrak{g}_\imath$ (resp. $\mathfrak{b}_\imath$) is the Lie algebra of $G_\imath$ (resp $B_\imath$). Hence, it suffices to determine the action of $T_\imath$ and σ on $\wedge^{d_\imath}(\mathfrak{g}_\imath/\mathfrak{b}_\imath)$, where $T_\imath$ is the maximal torus of $G_\imath$ contained in $B_\imath$. Note that

$$\wedge^{d_\imath}(\mathfrak{g}_\imath/\mathfrak{b}_\imath) \simeq \wedge^{d_\imath}\mathfrak{n}_\imath^-,$$

where $\mathfrak{n}_\imath^-$ is the nilpotent radical of the negative Borel subalgebra of $\mathfrak{g}_\imath$. Hence, as 1-dimensional representation of $T_\imath$, it is isomorphic to $-2\rho_\imath$, and by case-by-case analysis it is easy to check σ acts on it exactly by $\epsilon_\imath$. This finishes the proof of the lemma. □

Only when $\imath$ consists of two vertices and $\imath = \{i, j\}$ is not connected, $\epsilon_\imath = -1$; otherwise $\epsilon_\imath = 1$.

Lemma 4.3. *Given any $n \in \mathbb{Z}$ and any r-th root of unity ξ, there exists a unique isomorphism up to a scalar*

$$H^{d_\imath}(\mathcal{B}_\imath, \mathcal{L}(n\rho_\imath, \xi)) \simeq H^0(\mathcal{B}_\imath, \mathcal{L}((-n-2)\rho_\imath, \epsilon_\imath \cdot \xi)),$$

as representations of $G_\imath \rtimes \langle\sigma\rangle$. Moreover,

$$H^k(\mathcal{B}_\imath, \mathcal{L}(n\rho_\imath, \xi)) = 0 \quad \text{if } k \neq 0, d_\imath.$$

Proof. By Borel-Weil-Bott theorem we have the following isomorphism of representations of $G_\imath \rtimes \langle\sigma\rangle$

$$H^0(\mathcal{B}_\imath, \mathcal{L}(n\rho_\imath, \xi))^* = \begin{cases} V_{n\rho_\imath,\xi} & n \geq 0 \\ 0 & n < 0 \end{cases}, \tag{23}$$

for any $n \in \mathbb{Z}$ and r-th root of unity ξ, where $V_{n\rho_\imath,\xi}$ is the irreducible representation of $G_\imath$ of highest weight $n\rho_\imath$ with the compatible action of σ which acts on the highest weight vectors by ξ.

By Serre duality we have the following canonical isomorphism

$$H^{d_\imath}(\mathcal{B}_\imath, \mathcal{L}(n\rho_\imath, \xi)) \simeq H^0(\mathcal{B}_\imath, \mathcal{L}(-n\rho_\imath, \xi^{-1}) \otimes \Omega_{\mathcal{B}_\imath})^* \tag{24}$$

as representations of $G_\imath \rtimes \langle\sigma\rangle$. In view of Lemma 4.2,

$$H^0(\mathcal{B}_\imath, \mathcal{L}(-n\rho_\imath, \xi^{-1}) \otimes \Omega_{\mathcal{B}_\imath}) \simeq H^0(\mathcal{B}_\imath, \mathcal{L}((-n-2)\rho_\imath, \epsilon_\imath \cdot \xi^{-1})).$$

In view of (23), by Schur lemma there exists a unique isomorphism up to a scalar

$$H^0(\mathcal{B}_\imath, \mathcal{L}((-n-2)\rho_\imath, \epsilon_\imath \cdot \xi^{-1}))^* \simeq H^0(\mathcal{B}_\imath, \mathcal{L}((-n-2)\rho_\imath, \epsilon_\imath \cdot \xi)) \tag{25}$$

as representations of $G_\imath \rtimes \langle\sigma\rangle$. Therefore we have an isomorphism

$$H^{d_\imath}(\mathcal{B}_\imath, \mathcal{L}(n\rho_\imath, \xi)) \simeq H^0(\mathcal{B}_\imath, \mathcal{L}((-n-2)\rho_\imath, \epsilon_\imath \cdot \xi)) \tag{26}$$

as representations of $G_\imath \rtimes \langle\sigma\rangle$.

Now we prove the second part of the lemma. When $n \geq 0$, $n\rho_\imath$ is dominant, then Borel-Weil-Bott theorem implies that $H^k(\mathcal{B}_\imath, \mathcal{L}(n\rho_\imath, \xi)) = 0$ unless $k = 0$. In view of isomorphism (26), when $n \leq -2$, $H^k(\mathcal{B}_\imath, \mathcal{L}(n\rho_\imath, \xi)) = 0$ unless $k = d_\imath$. When $n = -1$, we have $s_i \star \rho_\imath = \rho_\imath$. Thus, $H^k(\mathcal{B}_\imath, \mathcal{L}(-\rho_\imath, \xi)) = 0$ for any k. $\square$

Let $\tilde{\mathcal{P}}_\imath$ be the parabolic subgroup of $\tilde{G}$ containing $\tilde{\mathcal{I}}$ and $G_\imath$. We have an isomorphism of varieties $\tilde{\mathcal{P}}_\imath/\tilde{\mathcal{I}} \simeq \mathcal{B}_\imath$. Let $\pi_\imath : \mathrm{Fl}_G \to \tilde{G}/\tilde{\mathcal{P}}_\imath$ be the projection map. The fiber is isomorphic to $\mathcal{B}_\imath$. There exists the following natural isomorphism as $\tilde{G} \rtimes \langle\sigma\rangle$-equivariant ind-schemes

$$\tilde{G} \rtimes \langle\sigma\rangle \times_{\tilde{\mathcal{P}}_\imath \rtimes \langle\sigma\rangle} \mathcal{B}_\imath \simeq \mathrm{Fl}_G.$$

From the $G_\imath \rtimes \langle\sigma\rangle$-equivariant line bundle $\mathcal{L}(n\rho_\imath, \xi)$ on $\mathcal{B}_\imath$, by descent theory one can attach a $\tilde{G} \rtimes \langle\sigma\rangle$-equivariant line bundle $\mathcal{L}_{\pi_\imath}(n\rho_\imath, \xi)$ on Fl_G, i.e.

$$\mathcal{L}_{\pi_\imath}(n\rho_\imath, \xi) := \tilde{G} \rtimes \langle\sigma\rangle \times_{\tilde{\mathcal{P}}_\imath \rtimes \langle\sigma\rangle} \mathcal{L}(n\rho_\imath, \xi),$$

where the action of $\tilde{\mathcal{P}}_\imath \rtimes \langle\sigma\rangle$ on $\mathcal{L}(n\rho_\imath, \xi)$ factors through $G_\imath \rtimes \langle\sigma\rangle$. Let $\Omega_{\pi_\imath}$ be the relative canonical line bundle of Fl_G over $\tilde{G}/\tilde{\mathcal{P}}_\imath$. By Lemma 4.2 as a $\tilde{G} \rtimes \langle\sigma\rangle$-equivariant bundle, we have

$$\Omega_{\pi_\imath} \simeq \mathcal{L}_{\pi_\imath}(-2\rho_\imath, \epsilon_\imath). \tag{27}$$

Let $R^k(\pi_\imath)_*$ be the k-th derived functor of the pushforward functor $(\pi_\imath)_*$. The following lemma is a relative version of Lemma 4.3.

Lemma 4.4. *There exists a natural isomorphism of $\tilde{G} \rtimes \langle\sigma\rangle$-equivariant vector bundles*

$$R^{d_\imath}(\pi_\imath)_*(\mathcal{L}_{\pi_\imath}(n\rho_\imath, \xi)) \simeq (\pi_\imath)_*(\mathcal{L}_{\pi_\imath}((-n-2)\rho_\imath, \xi \cdot \epsilon_\imath)).$$

Proof. By relative Serre duality for the morphism $\pi_\imath : \mathrm{Fl}_G \to \tilde{G}/\tilde{\mathcal{P}}_\imath$, there exists a canonical isomorphism of $\tilde{G} \rtimes \langle\sigma\rangle$-equivariant sheaves on $\tilde{G}/\tilde{\mathcal{P}}_\imath$,

$$R^{d_\imath}(\pi_\imath)_*(\mathcal{L}_{\pi_\imath}(n\rho_\imath, \xi)) \simeq (\pi_\imath)_*(\mathcal{L}_{\pi_\imath}(-n\rho_\imath, \xi^{-1}) \otimes \Omega_{\pi_\imath})^\vee,$$

where $\vee$ denotes the dual of coherent sheaf. From isomorphism (27), it gives rise to the following isomorphism of $\tilde{G} \rtimes \langle\sigma\rangle$-equivariant sheaves on $\tilde{G}/\tilde{\mathcal{P}}_\imath$,

$$R^{d_\imath}(\pi_\imath)_*(\mathcal{L}_{\pi_\imath}(n\rho_\imath, \xi)) \simeq (\pi_\imath)_*(\mathcal{L}_{\pi_\imath}((-n-2)\rho_\imath, \xi^{-1}\epsilon_\imath))^\vee. \tag{28}$$

We look at the fiber of the sheaf $(\pi_\imath)_*(\mathcal{L}_{\pi_\imath}((-n-2)\rho_\imath, \xi^{-1}\epsilon_\imath))^\vee$ at the base point $e\tilde{\mathcal{P}}_\imath \in \tilde{G}/\tilde{\mathcal{P}}_\imath$. This is the representation of $\tilde{\mathcal{P}}_\imath \ltimes \langle\sigma\rangle$ on $H^0(\mathcal{B}_\imath, \mathcal{L}((-n-2)\rho_\imath, \epsilon_\imath \cdot \xi^{-1}))^*$ by factoring through the map $\tilde{\mathcal{P}}_\imath \ltimes \langle\sigma\rangle \to G_\imath \ltimes \langle\sigma\rangle$. From isomorphism (25), the $\tilde{G} \rtimes \langle\sigma\rangle$-equivariance gives rise to an isomorphism of $\tilde{G} \rtimes \langle\sigma\rangle$-vector bundles

$$(\pi_\imath)_*(\mathcal{L}_{\pi_\imath}((-n-2)\rho_\imath, \xi^{-1}\epsilon_\imath))^\vee \simeq (\pi_\imath)_*(\mathcal{L}_{\pi_\imath}((-n-2)\rho_\imath, \xi \cdot \epsilon_\imath)).$$

Combining with (28), the lemma follows. $\square$

By Lemma 2.5, the affine Weyl group $(\hat{W})^\sigma$ consists of simple reflections $\{s_\imath \mid \imath \in \hat{I}_\sigma\}$.

Lemma 4.5. *For any σ-invariant weight Λ of $\tilde{\mathfrak{g}}$ and for any σ-orbit in $\hat{I}$, we have*

$$s_\imath \cdot \Lambda = \begin{cases} \Lambda - \langle \Lambda, \check{\alpha}_i \rangle \sum_{i \in \imath} \alpha_i & \text{if } a_{ij} = 0 \text{ for any } i \neq j \in \imath, \\ \Lambda - 2\langle \Lambda, \alpha_i \rangle (\alpha_i + \alpha_j) & \text{if } \imath = \{i, j\} \text{ is connected.} \end{cases}$$

Proof. For any σ-orbit $\imath$ in I, this is routine to check, in particular we use the formula (7). When $\imath = \{0\}$, this is simply the definition of s_0. $\square$

Proposition 4.6. *For any σ-invariant weight Λ, and for any σ-orbit in the affine Dynkin diagram $\hat{I}$ and any r-th root of unity ξ, we have the following isomorphism*

$$H^{k+d_\imath}(\mathrm{Fl}_G, \mathcal{L}(s_\imath \star \Lambda, \xi)) \simeq H^k(\mathrm{Fl}_G, \mathcal{L}(\Lambda, \epsilon_\imath \cdot \xi))$$

as representations of $\tilde{G} \rtimes \langle \sigma \rangle$, for all integer k.

Proof. Note that the restriction $\mathcal{L}(\Lambda, \xi)|_{\mathcal{B}_\imath}$ of the $\tilde{G} \rtimes \langle \sigma \rangle$-equivariant line bundle $\mathcal{L}(\Lambda, \xi)$ to the fiber $\mathcal{B}_\imath$ is isomorphic to $\mathcal{L}(\langle \Lambda, \check{\alpha}_i \rangle \rho_\imath, \xi)$ as a $G_\imath \rtimes \langle \sigma \rangle$-equivariant line bundle for any $i \in \imath$. Note that for any $i, j \in \imath$, $\langle \Lambda, \check{\alpha}_i \rangle = \langle \Lambda, \check{\alpha}_j \rangle$. In view of Lemma 4.5, we have

$$s_\imath \star \Lambda = \begin{cases} \Lambda - (\langle \Lambda, \check{\alpha}_i \rangle + 1) \sum_{i \in \imath} \alpha_i & \text{if } \imath \text{ is not connected} \\ \Lambda - 2(\langle \Lambda, \alpha_i \rangle + 1)(\alpha_i + \alpha_j) & \text{if } \imath = \{i, j\} \text{ is connected} \end{cases}.$$

Hence for any σ-orbit $\imath$ in $\hat{I}$ and $i \in \imath$, we have

$$\langle s_\imath \star \Lambda, \check{\alpha}_i \rangle = -\langle \Lambda, \check{\alpha}_i \rangle - 2.$$

It follows that

$$\mathcal{L}(s_\imath \star \Lambda, \xi)|_{\mathcal{B}_\imath} = \mathcal{L}(-(\langle \Lambda, \check{\alpha}_i \rangle + 2)\rho_\imath, \xi).$$

By Lemma 4.4, we have the following natural isomorphism of $\tilde{G} \rtimes \langle \sigma \rangle$-equivariant vector bundles

$$R^{d_\imath}\pi_*(\mathcal{L}(\Lambda, \xi)) \simeq \pi_*(\mathcal{L}(s_\imath \star \Lambda, \epsilon_\imath \cdot \xi). \tag{29}$$

By Lemma 4.3, we have

$$R^k\pi_*(\mathcal{L}(\Lambda, \xi)) = 0 \quad \text{if } k \neq 0, d_\imath. \tag{30}$$

In view of (29) and (30), Leray's spectral sequence implies that

$$H^{k+d_\imath}(\mathrm{Fl}_G, \mathcal{L}(\Lambda, \xi)) \simeq H^k(\mathrm{Fl}_G, \mathcal{L}(\Lambda, \epsilon_\imath \cdot \xi))$$

as representations of $\tilde{G} \rtimes \langle \sigma \rangle$. $\square$

For any $w \in (\hat{W})^\sigma$, put

$$\epsilon_w = (-1)^{\ell(w)-\ell_\sigma(w)}. \tag{31}$$

For any reduced expression $w = s_{\imath_k} s_{\imath_{k-1}} \cdots s_{\imath_1}$ of w in the Coxeter group $(\hat{W})^\sigma$ where $\imath_1, \cdots, \imath_k$ are σ-orbits in $\hat{I}$ and each $s_\imath$ is defined in (12) for any $\imath \in I_\sigma$ and $s_{\{0\}} = s_0$, we have $\epsilon_w = \epsilon_{\imath_k} \cdots \epsilon_{\imath_1}$, where $\epsilon_\imath$ is introduced in Lemma 4.2.

Finally, we are now ready to prove the following theorem.

Theorem 4.7. *For any $w \in (\hat{W})^\sigma$ and for any σ-invariant dominant weight Λ of $\tilde{G}$. We have the following isomorphism of representations of $\tilde{G} \rtimes \langle\sigma\rangle$*

$$H^{\ell(w)}(\mathrm{Fl}_G, \mathcal{L}(w \star \Lambda, \xi)) \simeq H^0(\mathrm{Fl}_G, \mathcal{L}(\Lambda, \epsilon_w \cdot \xi)).$$

Proof. We can write $w = s_{\imath_k} s_{\imath_{k-1}} \cdots s_{\imath_1}$ as a reduced expression in the Coxeter group $(\hat{W})^\sigma$, where $\imath_1, \cdots, \imath_k$ are σ-orbits in $\hat{I}$. Then

$$\Lambda, s_{\imath_1} \star \Lambda, (s_{\imath_2} s_{\imath_1}) \star \Lambda, \cdots, w \star \Lambda$$

are all σ-invariant weights of $\tilde{G}$.

Note that as an element in $\hat{W}$, the length $\ell(w)$ of w is equal to $\sum_{i=1}^k d_{\imath_i}$. In view of Proposition 4.6, we get a chain of isomorphisms of $\tilde{G} \rtimes \langle\sigma\rangle$-representations

$$\begin{aligned} H^{\ell(w)}(\mathrm{Fl}_G, \mathcal{L}(w \star \Lambda, \xi)) &\simeq H^{\ell(w)-d_{\imath_1}}(\mathrm{Fl}_G, \mathcal{L}((s_{\imath_1} w) \star \Lambda, \epsilon_{\imath_1}\xi)) \\ &\simeq H^{\ell(w)-d_{\imath_1}-d_{\imath_2}}(\mathrm{Fl}_G, \mathcal{L}((s_{\imath_2} s_{\imath_1} w) \star \Lambda, \epsilon_{\imath_2}\epsilon_{\imath_1}\xi)) \\ &\cdots\cdots \\ &\simeq H^0(\mathrm{Fl}_G, \mathcal{L}(\Lambda, \epsilon_w \cdot \xi)). \end{aligned}$$

This finishes the proof of the theorem. □

For any dominant weight Λ of $\tilde{\mathfrak{g}}$ and an r-th root of unity, as always we denote by $\mathcal{H}_{\Lambda,\xi}$ the irreducible integrable representation of $\tilde{\mathfrak{g}}$ of highest weight Λ together with a compatible action of σ which acts on the highest weight vectors of $\mathcal{H}_{\Lambda,\xi}$ by ξ.

Corollary 4.8. *In the same setting as in Theorem 4.7, we have the following isomorphism of representations of $\tilde{\mathfrak{g}} \rtimes \langle\sigma\rangle$,*

$$H^{\ell(w)}(\mathrm{Fl}_G, \mathcal{L}(w \star \Lambda, \xi))^\vee \simeq \mathcal{H}_{\Lambda,\epsilon_w\cdot\xi}.$$

Proof. This is an immediate consequence of Theorem 4.1 and Theorem 4.7. □

Remark 4.9. For any σ-invariant weight λ of G, let $\mathcal{L}(\lambda)$ be the associated line bundle on G/B. By Borel-Weil-Bott theorem, $H^i(G/B, \mathcal{L}(\lambda))$ carries an action of the diagram automorphism. The action was determined by Naito. Theorem 4.7 and Theorem 4.13 are the affine analogues of the results of Naito [28].

4.2. Borel-Weil-Bott theorem on affine Grassmannian

For any weight λ of G, let $\mathcal{L}_\ell(\lambda)$ be the $\hat{G}$-equivariant line bundle on Fl_G defined as follows,

$$\mathcal{L}_\ell(\lambda) := \hat{G} \times_{\hat{\mathcal{I}}} I_\ell(\mathbb{C}_\lambda)^*,$$

where $I_\ell(\mathbb{C}_\lambda)$ is the 1-dimensional representation of $\hat{\mathcal{I}}$ such that $\mathcal{I}$ factors through the character $\lambda : B \to \mathbb{C}^\times$ and the center $\mathbb{C}^\times$ acts by $t \mapsto t^\ell$, and $I_\ell(\lambda)^*$ is the dual of $I_\ell(\lambda)$ as the representation of $\mathcal{I}$.

For any character Λ of $\tilde{\mathcal{I}}$, if $\Lambda = \lambda + \ell\Lambda_0$ where Λ is a weight of $\tilde{G}$ and λ is a weight of G, then as $\hat{G}$-equivariant line bundles, $\mathcal{L}(\Lambda) = \mathcal{L}_\ell(\lambda)$.

If λ is σ-invariant, then $\mathcal{L}_\ell(\lambda)$ has a natural σ-equivariant structure as in the case of $\mathcal{L}(\Lambda)$. Similarly, to an r-th root of unity ξ where r is the order of σ, we can associate a $\hat{G} \rtimes \langle\sigma\rangle$-equivariant line bundle $\mathcal{L}_\ell(\lambda, \xi)$. If $\Lambda = \lambda + \ell\Lambda_0$ where $\lambda \in P^\sigma$, then $\mathcal{L}(\Lambda, \xi) = \mathcal{L}_\ell(\lambda, \xi)$ as $\hat{G} \rtimes \langle\sigma\rangle$-equivariant line bundles.

Recall from Lemma 3.3, the weight $\Lambda = \lambda + \ell\Lambda_0$ is dominant for $\tilde{G}$ if and only if λ is dominant for G and $\langle\lambda, \check{\theta}\rangle \leq \ell$. Recall the affine Weyl group $W_{\ell+\check{h}}$ discussed in Section 2.2, the action of $W_{\ell+\check{h}}$ on the weight lattice P of G is compatible with the action of $\hat{W}$ on the space of weights of $\tilde{G}$ of level ℓ, see Lemma 3.1 and Lemma 3.2. Therefore we can translate Theorem 4.7 into the following equivalent theorem.

Theorem 4.10. *For any $w \in W_{\ell+\check{h}}$ such that $\sigma(w) = w$ and for any σ-invariant dominant weight $\lambda \in P_\ell$, we have the following isomorphism*

$$H^{\ell(w)}(\mathrm{Fl}_G, \mathcal{L}_\ell(w \star \lambda, \xi)) \simeq H^0(\mathrm{Fl}_G, \mathcal{L}_\ell(\lambda, \epsilon_w \cdot \xi))$$

as representations of $\hat{G} \rtimes \langle\sigma\rangle$.

Let $\hat{\mathcal{P}}$ be the subgroup $G[[t]] \times \mathbb{C}^\times$ of $\hat{G}$ where $\mathbb{C}^\times$ is the center torus. The affine Grassmannian $\mathrm{Gr}_G := G((t))/G[[t]]$ is isomorphic to the partial flag variety $\hat{G}/\hat{\mathcal{P}}$. For any finite dimensional representation V of G, let $I_\ell(V)$ be the representation of $\hat{\mathcal{P}}$ such that $G[[t]]$ acts via the evaluation map $\mathrm{ev}_0 : G[[t]] \to G$ given by evaluating $t = 0$, and the center $\mathbb{C}^\times$ acts by $t \mapsto t^\ell$. Let $\mathcal{L}_\ell(V)$ be the induced $\hat{G}$-equivariant vector bundle on Gr_G, i.e. $\mathcal{L}_\ell(V) := \hat{G} \times_{\hat{\mathcal{P}}} I_\ell(V)^*$, where $I_\ell(V)^*$ is the dual of $I_\ell(V)$ as the representation of $\hat{\mathcal{P}}$.

The diagram automorphism σ on G induces an automorphism on $\hat{G}$ and it preserves $\hat{\mathcal{P}}$. For any $\lambda \in (P^+)^\sigma$, the vector bundle $\mathcal{L}_\ell(V_\lambda)$ is naturally equipped with a σ-equivariant structure, since

$$\hat{G} \rtimes \langle\sigma\rangle \times_{\hat{\mathcal{P}} \rtimes \langle\sigma\rangle} I_\ell(V_\lambda)^* \simeq \hat{G} \times_{\hat{\mathcal{P}}} I_\ell(V_\lambda)^*.$$

Similarly, for any r-th root of unity ξ, we have the $\hat{G} \rtimes \langle\sigma\rangle$-equivariant vector bundle $\mathcal{L}_\ell(V_{\lambda,\xi})$ on Gr_G.

The following lemma is well-known.

Lemma 4.11. *Let H_1 be a linear algebraic group and H_2 be a subgroup of H_1. Let V_1 be a finite dimensional representation of H_1 and let V_2 be a finite dimensional representation of H_2. Then we have an isomorphism of H_1-equivariant vector bundles*

$$H_1 \times_{H_2} (V_2 \otimes V_1|_{H_2}) \simeq (H_1 \times_{H_2} V_2) \otimes V_1,$$

given by $(h_1, v_2 \otimes v_1) \mapsto (h_1, v_2) \otimes h_1 \cdot v_1$, where $h_1 \in H_1$, $v_1 \in V_1$ and $v_2 \in V_2$.

Lemma 4.12. *Let λ be a σ-invariant dominant weight of G, and let V be a finite dimensional representation of $G \rtimes \langle\sigma\rangle$. There is an isomorphism of $\hat{G} \rtimes \langle\sigma\rangle$-representations*

$$H^k(\mathrm{Gr}_G, \mathcal{L}_\ell(V_{\lambda,\xi} \otimes V)) \simeq H^k(\mathrm{Fl}_G, \mathcal{L}_\ell(\mathbb{C}_{\lambda,\xi} \otimes V|_{B \rtimes \langle\sigma\rangle}),$$

for any $k \geq 0$ and ξ an r-th root of unity.

Proof. We have the following isomorphisms of $\hat{G} \rtimes \langle\sigma\rangle$-equivariant vector bundles

$$\begin{aligned}
\mathcal{L}_\ell(\mathbb{C}_{\lambda,\xi} \otimes V|_{B \rtimes \langle\sigma\rangle}) &\simeq \hat{G} \rtimes \langle\sigma\rangle \times_{\hat{\mathcal{I}} \rtimes \langle\sigma\rangle} (\mathbb{C}_{\lambda,\xi} \otimes V|_{B \rtimes \langle\sigma\rangle})^* \\
&\simeq \hat{G} \rtimes \langle\sigma\rangle \times_{\hat{\mathcal{P}} \rtimes \langle\sigma\rangle} (\hat{\mathcal{P}} \rtimes \langle\sigma\rangle \times_{\hat{\mathcal{I}} \rtimes \langle\sigma\rangle} I_\ell(\mathbb{C}_{\lambda,\xi} \otimes V|_{B \rtimes \langle\sigma\rangle}))^* \\
&\simeq \hat{G} \rtimes \langle\sigma\rangle \times_{\hat{\mathcal{P}} \rtimes \langle\sigma\rangle} ((\hat{\mathcal{P}} \rtimes \langle\sigma\rangle \times_{\hat{\mathcal{I}} \rtimes \langle\sigma\rangle} I_\ell(\mathbb{C}_{\lambda,\xi})) \otimes V)^*,
\end{aligned}$$

where the last isomorphism follows from Lemma 4.11.

It is a $\hat{G} \rtimes \langle\sigma\rangle$-equivariant vector bundle on Fl_G. By Borel-Weil-Bott theorem for finite type algebraic group, we have

$$R^k \pi_* \mathcal{L}_\ell(\mathbb{C}_{\lambda,\xi} \otimes V|_{B \rtimes \langle\sigma\rangle}) \simeq \begin{cases} 0 & k > 0 \\ \mathcal{L}_\ell(V_{\lambda,\xi} \otimes V) & k = 0 \end{cases},$$

where $R^k\pi_*$ is the right derived functor of π_*. By Leray's spectral sequence, the lemma follows. $\square$

Let $W^\dagger_{\ell+\check{h}}$ denote the set of the minimal representatives of the left cosets of W in $W_{\ell+\check{h}}$, then for any $w_1 \in W$ and $w_2 \in W^\dagger_{\ell+\check{h}}$, we have $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$. Moreover for any $w \in W_{\ell+\check{h}}$ and $\lambda \in P_\ell$

$$w \star \lambda \in P^+ \text{ if and only if } w \in W^\dagger_{\ell+\check{h}}, \tag{32}$$

see [20, Remark 1.3]. Since P_ℓ is the set of integral points in the fundamental alcove of the affine Weyl group $W_{\ell+\check{h}}$, for any dominant weight $\lambda \in P^+$, there exists a unique $w \in W^\dagger_{\ell+\check{h}}$ such that $w^{-1} \star \lambda \in P_\ell$. By Lemma 2.6, for any σ-invariant dominant weight $\lambda \in P^+$, there exists a unique $w \in (W^\dagger_{\ell+\check{h}})^\sigma$ such that $w^{-1} \star \lambda \in P^\sigma_\ell$.

Recall that we defined in Section 3.3 the representation $V_{\lambda,\xi}$ of $\mathfrak{g} \rtimes \langle\sigma\rangle$ as the representation V_λ of $\mathfrak{g}$ together with an operator σ such that σ acts on the highest weight vectors by ξ, where $\lambda \in (P^+)^\sigma$ and ξ is an r-th root of unity. Similarly, the representation $\mathcal{H}_{\lambda,\xi}$ is the representation $\mathcal{H}_\lambda$ of $\hat{\mathfrak{g}} \rtimes \langle\sigma\rangle$ of level ℓ together with an operator σ such that σ acts on the highest weight vectors by ξ. We have the following theorem

Theorem 4.13. *For any $w \in W^\dagger_{\ell+\check{h}}$ such that $\sigma(w) = w$ and for any $\lambda \in P^\sigma_\ell$, we have the following isomorphism of representations of $\hat{G} \rtimes \langle\sigma\rangle$,*

$$H^{\ell(w)}(\mathrm{Gr}_G, \mathcal{L}_\ell(V_{w\star\lambda,\xi})) \simeq H^0(\mathrm{Gr}_G, \mathcal{L}_\ell(V_{\lambda,\epsilon_w\xi})).$$

Proof. This follows from Theorem 4.10 and Lemma 4.12. $\square$

Corollary 4.14. *With the same assumption as in Theorem 4.13.*

(1) *There exists an isomorphism of representations of $\hat{\mathfrak{g}} \rtimes \langle\sigma\rangle$*

$$H^{\ell(w)}(\mathrm{Gr}_G, \mathcal{L}_\ell(V_\lambda))^\vee \simeq \mathcal{H}_{\lambda,\epsilon_w}.$$

(2) *There exists an isomorphism of representations of $\mathfrak{g} \rtimes \langle\sigma\rangle$*

$$(H^{\ell(w)}(\mathrm{Gr}_G, \mathcal{L}_\ell(V_{w\star\lambda}))^\vee)_{\hat{\mathfrak{g}}^-} \simeq V_{\lambda,\epsilon_w},$$

where $\hat{\mathfrak{g}}^- = t^{-1}\mathfrak{g}[t^{-1}]$.

Proof. This proposition follows from Theorem 4.13, combining with Corollary 4.8, Lemma 4.12 and Lemma 3.4. $\square$

4.3. Affine analogues of BBG resolution and Kostant homology

We first recall the construction of BGG resolution in the setting of affine Lie algebra, we refer the reader to [21, Section 9.1] for more details, in particular Theorem 9.1.3 therein. There exists a Koszul resolution of the trivial representation $\mathbb{C}$ of $\hat{\mathfrak{g}}$,

$$\cdots \to X_p \xrightarrow{\delta_p} \cdots \xrightarrow{\delta_1} X_0 \xrightarrow{\delta_0} \mathbb{C},$$

where $X_p = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{p}})} \wedge^p(\hat{\mathfrak{g}}/\hat{\mathfrak{p}})$. From the construction of Koszul resolution, this complex is $\hat{\mathfrak{g}} \rtimes \langle\sigma\rangle$-equivariant. Given a σ-invariant dominant weight $\lambda \in P_\ell$. Set $X_{\lambda,p} := U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{p}})} (\wedge^p(\hat{\mathfrak{g}}/\hat{\mathfrak{p}}) \otimes \mathcal{H}_\lambda)$. The complex $X_{\lambda,\bullet}$ is a resolution of $\mathcal{H}_\lambda$. Set

$$F_{\lambda,p} := \bigoplus_{w \in W^\dagger_{\ell+\check{h}}, \ell(w)=p} \hat{M}(V_{w\star\lambda}), \tag{33}$$

where $\hat{M}(V_{w\star\lambda})$ is the generalized Verma module introduced in Section 3.3. In fact $F_{\lambda,\bullet}$ is a σ-stable subcomplex of $X_{\lambda,\bullet}$, and moreover $X_{\lambda,\bullet}$ is quasi-isomorphic to $F_{\lambda,\bullet}$. Hence $F_{\lambda,\bullet}$ is a resolution of $\mathcal{H}_\lambda$.

The proof of the following proposition heavily replies on the work of Naito [29].

Proposition 4.15. *Assume that $\sigma(\lambda) = \lambda$. Then the complex $F_{\lambda,\bullet}$ is a resolution of $\mathcal{H}_\lambda$ as representations of $\hat{\mathfrak{g}} \rtimes \langle\sigma\rangle$, where σ maps $\hat{M}(V_{w\star\lambda})$ to $\hat{M}(V_{\sigma(w)\star\lambda})$. In particular when $\sigma(w) = w$, σ acts on the highest weight vectors of $\hat{M}(V_{w\star\lambda})$ by the scalar ϵ_w, where $\epsilon_w = (-1)^{\ell(w)-\ell_\sigma(w)}$ as defined in (31).*

Proof. First of all, we note that σ maps $\hat{M}(V_{w\star\lambda})$ to $\hat{M}(V_{\sigma(w)\star\lambda})$ for any $w \in W^\dagger_{\ell+\hat{h}}$, since $\sigma(\rho) = \rho$. In particular if $\sigma(w) = w$, σ keeps $\hat{M}(w\star\lambda)$ stable. We need to determine the action of σ at the highest weight vector $m_{w\star\lambda}$ of $\hat{M}(w \star \lambda)$. It is easy to see that σ acts on $m_{w\star\lambda}$ by a scalar ϵ'_w. In the following we will show that $\epsilon'_w = \epsilon_w$.

Recall that $\hat{\mathfrak{g}}^-$ denote the nilpotent Lie algebra $t^{-1}\mathfrak{g}[t^{-1}]$. It is standard that $\hat{M}(V_{w\star\lambda})$ is a free $U(\hat{\mathfrak{g}}^-)$-module, for each $w \in W^\dagger_{\ell+\check{h}}$. Thus, the resolution $F_{\lambda,\bullet}$ can be used to compute the $\hat{\mathfrak{g}}^-$-homologies of $\mathcal{H}_\lambda$, in other words,

$$H_p(\hat{\mathfrak{g}}^-, \mathcal{H}_\lambda) \simeq H_p((F_{\lambda,\bullet})_{\hat{\mathfrak{g}}^-}), \tag{34}$$

where the LHS is the p-th $\hat{\mathfrak{g}}^-$-homology of $\mathcal{H}_\lambda$, and the RHS is the p-th homology of the complex $(F_{\lambda,\bullet})_{\hat{\mathfrak{g}}^-}$ obtained from taking $\hat{\mathfrak{g}}^-$-coinvariants on the complex $F_{\lambda,\bullet}$. Moreover, the isomorphism (34) is $\mathfrak{g} \rtimes \langle\sigma\rangle$-equivariant. As a consequence, we get the following isomorphism of $\mathfrak{g} \rtimes \langle\sigma\rangle$-representations,

$$H_p(\hat{\mathfrak{g}}^-, \mathcal{H}_\lambda) \simeq \bigoplus_{w \in W^\dagger_{\ell+\check{h}}, \ell(w)=p} V_{w\star\lambda} \tag{35}$$

for each $p \geq 0$, since $(\hat{M}(V_{w\star\lambda}))_{\hat{\mathfrak{g}}^-} \simeq V_{w\star\lambda}$ as representations of $\mathfrak{g}$ (cf. Lemma 3.4). As mentioned above, σ acts on $m_{w\star\lambda} \in \hat{M}(V_{w\star\lambda})$ by the scalar ϵ'_w if $\sigma(w) = w$. It follows that σ acts on the highest weight vector $v_{w\star\lambda}$ of $V_{w\star\lambda}$ by ϵ'_w if $\sigma(w) = w$.

Let $\mathfrak{n}^-$ be the nilpotent radical of the negative Borel subalgebra $\mathfrak{b}^-$ of $\mathfrak{g}$. Put

$$\hat{\mathfrak{n}}^- := \hat{\mathfrak{g}}^- \oplus \mathfrak{n}^-.$$

Note that $\hat{\mathfrak{n}}^-$ is the nilpotent radical of the opposite affine Borel subalgebra $\hat{\mathfrak{b}}^- := \hat{\mathfrak{g}}^- \oplus \mathfrak{b}^-$ of $\hat{\mathfrak{g}}$, and $\hat{\mathfrak{n}}^-$ is σ-stable. Since $\hat{\mathfrak{g}}^-$ is an ideal in the Lie algebra $\hat{\mathfrak{n}}^-$, we have the following spectral sequence which is compatible with the actions of σ,

$$H_i(\mathfrak{n}^-, H_j(\hat{\mathfrak{g}}^-, \mathcal{H}_\lambda)) \Rightarrow H_{i+j}(\hat{\mathfrak{n}}^-, \mathcal{H}_\lambda). \tag{36}$$

Meanwhile, $H_i(\mathfrak{n}^-, H_j(\hat{\mathfrak{g}}^-, \mathcal{H}_\lambda))$ and $H_{i+j}(\hat{\mathfrak{n}}^-, \mathcal{H}_\lambda)$ both carry the actions of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}^-$. In fact the spectral sequence (36) degenerates at E_1 page, since we have the following sequence of isomorphisms of $\mathfrak{h}$-modules:

$$\begin{aligned}
\bigoplus_{i+j=p} H_i(\mathfrak{n}^-, H_j(\hat{\mathfrak{g}}^-, \mathcal{H}_\lambda)) &\simeq \bigoplus_{i+j=p} \bigoplus_{w \in W^\dagger_{\ell+\check{h}}, \ell(w)=j} H_i(\mathfrak{n}^-, V_{w\star\lambda}) \\
&\simeq \bigoplus_{i+j=p} \bigoplus_{w \in W^\dagger_{\ell+\check{h}}, \ell(w)=j} \bigoplus_{y \in W, \ell(y)=i} \mathbb{C}_{y\star(w\star\lambda)} \\
&\simeq \bigoplus_{w \in W_{\ell+\check{h}}, \ell(w)=p} \mathbb{C}_{w\star\lambda} \\
&\simeq H_p(\hat{\mathfrak{n}}^-, \mathcal{H}_\lambda),
\end{aligned}$$

where the first isomorphism follows from (34), the second isomorphism follows from Kostant homology formula for $\mathfrak{n}^-$, the last isomorphism follows from the affine version of Kostant homology formula for $\hat{\mathfrak{n}}^-$ (cf. [8]), and the third isomorphism follows since $W^\dagger_{\ell+\check{h}}$ is the set of minimal representatives of the left cosets of W in $W_{\ell+\check{h}}$. The set $W^\dagger_{\ell+\check{h}}$ satisfies the following property: for any $u \in W_{\ell+\check{h}}$, there exist unique $w \in W^\dagger_{\ell+\check{h}}$ and $y \in W$ such that $u = yw$ and $\ell(u) = \ell(y) + \ell(w)$.

We now make a digression on twining characters. Let V be a finite dimensional $\mathfrak{h} \rtimes \langle\sigma\rangle$-representation such that $\mathfrak{h}$ acts on V semi-simply. Define

$$\mathrm{ch}_\sigma(V) := \sum_{\mu \in \mathfrak{h}^*, \sigma(\mu)=\mu} \mathrm{tr}(\sigma | V(\mu)) e^\mu,$$

where $V(\mu)$ denotes the μ-weight space in V. Then

$$\sum_{i+j=p} \mathrm{ch}_\sigma(H_i(\mathfrak{n}^-, H_j(\hat{\mathfrak{g}}^-, \mathcal{H}_\lambda))) = \sum_{i+j=p} \sum_{w \in (W^\dagger_{\ell+\check{h}})^\sigma, \ell(w)=j} \epsilon'_w \mathrm{ch}_\sigma(H_i(\mathfrak{n}^-, V_{w\star\lambda})) \tag{37}$$

$$= \sum_{i+j=p} \sum_{w \in (W^\dagger_{\ell+\check{h}})^\sigma, \ell(w)=j} \epsilon'_w \sum_{y \in W^\sigma, \ell(y)=i} c_y(\sigma, V_{w\star\lambda}) e^{y\star(w\star\lambda)} \tag{38}$$

$$= \sum_{i+j=p} \sum_{\substack{w \in (W^\dagger_{\ell+\check{h}})^\sigma, \ell(w)=j \\ y \in W^\sigma, \ell(y)=i}} \epsilon'_w c_y(\sigma, V_{w\star\lambda}) e^{(yw)\star\lambda}, \tag{39}$$

where $c_y(\sigma, V_{w\star\lambda}) := \mathrm{tr}(\sigma|H_i(\mathfrak{n}^-, V_{w\star\lambda})_{(yw)\star\lambda})$. Here $H_i(\mathfrak{n}^-, V_{w\star\lambda})_{(yw)\star\lambda}$ denotes the $(yw)\star\lambda$-weight space in $H_i(\mathfrak{n}^-, V_{w\star\lambda})$. In the above sequence of equalities, the first equality follows from (35) and the discussions after that, the second equality follows from [29, Prop.3.2.1] for the Kostant homology of $\mathfrak{n}^-$. By [29, Prop.3.2.1] for the Kostant homology of $\hat{\mathfrak{n}}^-$, we have

$$\mathrm{ch}_\sigma(H_p(\hat{\mathfrak{n}}^-, \mathcal{H}_\lambda)) = \sum_{u\in(W_{\ell+\check{h}})^\sigma, \ell(u)=p} c_u(\sigma, \mathcal{H}_\lambda)e^{u\star\lambda}, \tag{40}$$

where $c_u(\sigma, \mathcal{H}_\lambda) := \mathrm{tr}(\sigma|H_p(\hat{\mathfrak{n}}^-, \mathcal{H}_\lambda)_{u\star\lambda})$. Here $H_p(\hat{\mathfrak{n}}^-, \mathcal{H}_\lambda)_{u\star\lambda}$ denotes the $u\star\lambda$-weight space in $H_p(\hat{\mathfrak{n}}^-, \mathcal{H}_\lambda)$. Since the spectral sequence (36) degenerates at E_2, we have

$$\sum_{i+j=p} \mathrm{ch}_\sigma(H_i(\mathfrak{n}^-, H_j(\hat{\mathfrak{g}}^-, \mathcal{H}_\lambda))) = \mathrm{ch}_\sigma(H_p(\hat{\mathfrak{n}}^-, \mathcal{H}_\lambda)). \tag{41}$$

Comparing formulae (39) and (40) via (41), we see that for any $w \in (W^\dagger_{\ell+\check{h}})^\sigma$, $c_w(\sigma, \mathcal{H}_\lambda) = \epsilon'_w c_e(\sigma, V_{w\star\lambda})$, where e is the identity element in the Weyl group W. Clearly $c_e(\sigma, V_{w\star\lambda}) = 1$, hence $c_w(\sigma, \mathcal{H}_\lambda) = \epsilon'_w$. We can read further from [29, Corollary 3.2.3], in fact $c_w(\sigma, \mathcal{H}_\lambda) = \epsilon_w$. Hence $\epsilon'_w = \epsilon_w$. Thus, this finishes the proof. $\square$

For any finite dimensional representation V of $\mathfrak{g}$ and for any $z \in \mathbb{C}^\times$, we denote by V^z the representation of $\hat{\mathfrak{g}}^-$ that is obtained by evaluating t at z. Let $H_i(\hat{\mathfrak{g}}^-, \mathcal{H}_\lambda \otimes V^z_\mu)$ be the i-th $\hat{\mathfrak{g}}^-$-homology on $\mathcal{H}_\lambda \otimes V^z_\mu$ where $\hat{\mathfrak{g}}^-$ acts on $\mathcal{H}_\lambda \otimes V^z_\mu$ diagonally. The following theorem will be used in the proof of Theorem 5.10.

Theorem 4.16. *For any $\lambda \in P^\sigma_\ell$ and $\mu \in (P^+)^\sigma$, the $\hat{\mathfrak{g}}^-$-homology groups $H_*(\hat{\mathfrak{g}}^-, \mathcal{H}_\lambda \otimes V^1_\mu)$ can be computed by the cohomology groups of a complex of $\mathfrak{g} \rtimes \langle\sigma\rangle$-representations,*

$$\cdots \to D_p \xrightarrow{\delta^p} \cdots D_1 \xrightarrow{\delta^1} D_0 \xrightarrow{\delta^0} 0,$$

where as representations of $\mathfrak{g}$, $D_p = \bigoplus_{w\in W^\dagger_{\ell+\check{h}},\, \ell(w)=p} V_{w\star\lambda} \otimes V_\mu$, and σ maps $V_{w\star\lambda} \otimes V_\mu$ to $V_{\sigma(w)\star\lambda} \otimes V_\mu$. In particular if $\sigma(w) = w$, then σ acts on the highest weight vectors of $V_{w\star\lambda}$ by $\epsilon_w = (-1)^{\ell(w)-\ell_\sigma(w)}$.

Proof. From the resolution $F_{\lambda,\bullet} \to \mathcal{H}_\lambda$, by tensoring with V^1_μ we get a resolution of $\mathcal{H}_\lambda \otimes V^1_\mu$ as representations of $\hat{\mathfrak{g}} \rtimes \langle\sigma\rangle$

$$\cdots \to F_{\lambda,p} \otimes V^1_\mu \xrightarrow{\delta^p} \cdots F_{\lambda,1} \otimes V^1_\mu \xrightarrow{\delta^1} F_{\lambda,0} \otimes V^1_\mu \xrightarrow{\delta^0} 0.$$

As $\mathfrak{g}$-modules, we have

$$(\hat{M}(V_{w\star\lambda}) \otimes V^1_\mu)_{\hat{\mathfrak{g}}^-} \simeq (V_{w\star\lambda} \otimes_{\mathbb{C}} U(\hat{\mathfrak{g}}^-)) \otimes_{U(\hat{\mathfrak{g}}^-)} V^1_\mu \simeq V_{w\star\lambda} \otimes V_\mu.$$

Hence the complex

$$\cdots \to (F_{\lambda,p} \otimes V^1_\mu)_{\hat{\mathfrak{g}}^-} \xrightarrow{\delta^p} \cdots (F_{\lambda,1} \otimes V^1_\mu)_{\hat{\mathfrak{g}}^-} \xrightarrow{\delta^1} (F_{\lambda,0} \otimes V^1_\mu)_{\hat{\mathfrak{g}}^-} \xrightarrow{\delta^0} 0$$

is quasi-isomorphic to

$$\cdots \to D_p \xrightarrow{\delta^p} \cdots D_1 \xrightarrow{\delta^1} D_0 \xrightarrow{\delta^0} 0.$$

By Proposition 4.15, σ maps $V_{w\star\lambda} \otimes V_\mu$ to $V_{\sigma(w)\star\lambda} \otimes V_\mu$. In particular if $\sigma(w) = w$, then σ acts on the highest weight vectors of $V_{w\star\lambda}$ by $\epsilon_w = (-1)^{\ell(w)-\ell_\sigma(w)}$. $\square$

5. σ-twisted representation ring and fusion ring

5.1. σ-twisted representation ring

Let V be a finite dimensional representation of $\mathfrak{g}$. For any irreducible representation V_λ of $\mathfrak{g}$ of highest weight λ, we denote by $\mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V)$ the multiplicity space of V_λ in V. In particular we have the following natural decomposition

$$V = \bigoplus_{\lambda \in P^+} \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V) \otimes V_\lambda.$$

Let $R(\mathfrak{g}, \sigma)$ be the free abelian group with the symbols $[V_\lambda]_\sigma$ as a basis, where $\lambda \in (P^+)^\sigma$. Given any finite dimensional representation V of $\mathfrak{g} \rtimes \langle\sigma\rangle$, V can be decomposed as follows

$$V = \bigoplus_{\lambda \in (P^+)^\sigma} \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V) \otimes V_\lambda \oplus \bigoplus_{\lambda \notin (P^+)^\sigma} \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V) \otimes V_\lambda,$$

as a representation of $\mathfrak{g}$. Put

$$[V]_\sigma := \sum_{\lambda \in (P^+)^\sigma} \mathrm{tr}(\sigma | \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V))[V_\lambda]_\sigma \in R(\mathfrak{g}, \sigma).$$

Let X be a finite dimensional representation of the cyclic group $\langle\sigma\rangle$, and for any representation V of $\mathfrak{g} \rtimes \langle\sigma\rangle$, $X \otimes V$ is naturally a representation of $\mathfrak{g} \rtimes \langle\sigma\rangle$, which is defined as follows

$$(u, \sigma^i) \cdot x \otimes v = \sigma^i \cdot x \otimes (u, \sigma^i) \cdot v,$$

where $u \in \mathfrak{g}, x \in X, v \in V$ and $i \in \mathbb{Z}$. Similarly, $V \otimes X$ is also naturally a representation of $\mathfrak{g} \rtimes \langle\sigma\rangle$. The following lemma is obvious.

Lemma 5.1. *We have* $[X \otimes V]_\sigma = \mathrm{tr}(\sigma|X)[V]_\sigma$*, and* $[V \otimes X]_\sigma = \mathrm{tr}(\sigma|X)[V]_\sigma$*.*

We define a multiplication $\otimes$ on $R(\mathfrak{g},\sigma)$, $[V_\lambda]_\sigma \otimes [V_\mu]_\sigma := [V_\lambda \otimes V_\mu]_\sigma$, for any $\lambda, \mu \in (P^+)^\sigma$. By definition, we have

$$[V_\lambda \otimes V_\mu]_\sigma = \sum_{\sigma(\nu)=\nu} \mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_\nu, V_\lambda \otimes V_\mu))[V_\nu]_\sigma.$$

Proposition 5.2. *$R(\mathfrak{g},\sigma)$ is a commutative ring with $[V_0]_\sigma$ as the unit.*

Proof. The commutativity is clear. We first show that the product $\otimes$ on $R(\mathfrak{g},\sigma)$ satisfies the associativity, i.e. for any $\lambda, \mu, \nu \in (P^+)^\sigma$,

$$([V_\lambda]_\sigma \otimes [V_\mu]_\sigma) \otimes [V_\nu]_\sigma = [V_\lambda]_\sigma \otimes ([V_\mu]_\sigma \otimes [V_\nu]_\sigma).$$

It suffices to show that for any $\lambda \in (P^+)^\sigma$ and any representation V of $\mathfrak{g} \rtimes \langle\sigma\rangle$,

$$[V_\lambda]_\sigma \otimes [V]_\sigma = [V_\lambda \otimes V]_\sigma, \text{ and } [V]_\sigma \otimes [V_\lambda]_\sigma = [V \otimes V_\lambda]_\sigma.$$

We have the following equalities

$$\begin{aligned}
[V_\lambda]_\sigma \otimes [V]_\sigma &= \sum_{\sigma(\mu)=\mu} \mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_\mu, V))([V_\lambda]_\sigma \otimes [V_\mu]_\sigma) \\
&= \sum_{\sigma(\mu)=\mu} \mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_\mu, V))[V_\lambda \otimes V_\mu]_\sigma \\
&= \sum_{\sigma(\mu)=\mu} [V_\lambda \otimes V_\mu \otimes \mathrm{Hom}_{\mathfrak{g}}(V_\mu, V)]_\sigma \\
&= [\bigoplus_\mu V_\lambda \otimes V_\mu \otimes \mathrm{Hom}_{\mathfrak{g}}(V_\mu, V)]_\sigma = [V_\lambda \otimes V]_\sigma,
\end{aligned}$$

where the third equality follows from Lemma 5.1, and others follows from definition of the multiplication $\otimes$. The equality $[V]_\sigma \otimes [V_\lambda]_\sigma = [V \otimes V_\lambda]_\sigma$ can be proved similarly. In the end $[V_0]_\sigma$ is the unit since for any $\lambda \in (P^+)^\sigma$,

$$[V_\lambda]_\sigma \otimes [V_0]_\sigma = [V_0]_\sigma \otimes [V_\lambda]_\sigma = [V_\lambda \otimes V_0]_\sigma = [V_\lambda]_\sigma. \quad \square$$

Recall that W_λ denotes the representation of $\mathfrak{g}_\sigma$ of highest weight $\iota(\lambda)$, and $W_\lambda(\mu)$ is the $\iota(\mu)$-weight space of W_λ, where ι is defined in Section 2.1. The following theorem is due to Jantzen [18].

Theorem 5.3. *Let $\lambda \in (P^+)^\sigma$ and $\mu \in P^\sigma$. We have $\mathrm{tr}(\sigma|V_\lambda(\mu)) = \dim W_\lambda(\mu)$.*

For any finite dimensional representation V of $\mathfrak{g} \rtimes \langle\sigma\rangle$, we define the σ-twisted character $\mathrm{ch}_\sigma(V)$ of V as follows

$$\mathrm{ch}_\sigma(V) := \sum_{\mu \in P^\sigma} \mathrm{tr}(\sigma|V(\mu))e^\mu,$$

where $V(\mu)$ denotes the μ-weight space of V. The following lemma is obvious.

Lemma 5.4. *For any two finite dimensional $\mathfrak{g} \rtimes \langle\sigma\rangle$-representations V, V', we have*

$$\mathrm{ch}_\sigma(V \otimes V') = \mathrm{ch}_\sigma(V)\mathrm{ch}_\sigma(V').$$

Lemma 5.5. *Let $\vec{\lambda}$ be a tuple of σ-invariant dominant weights of $\mathfrak{g}$ and let ν be another σ-invariant dominant weight of $\mathfrak{g}$. The following equality holds*

$$\mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_\nu, V_{\vec{\lambda}})) = \mathrm{tr}(\sigma|(V_{\vec{\lambda}} \otimes V_{\nu^*})^{\mathfrak{g}}).$$

Proof. Let w_0 be the longest element in the Weyl group W of $\mathfrak{g}$. There exists a representative $\bar{w}_0$ of w_0 in G such that $\sigma(\bar{w}_0) = \bar{w}_0$ (see [13, Section 2.3]). Hence $\sigma(\overline{w}_0 \cdot v_\nu) = \overline{w}_0 \cdot v_\nu$, where $v_\nu \in V_\nu$ is the highest weight vector. The vector $\overline{w}_0 \cdot v_\nu$ is of the lowest weight $w_0(\nu)$. Let V_ν^* denote the dual representation of V_ν, and let σ^* be the action on V_ν^* induced by the action σ on V_ν. Then σ^* keeps the highest weight vectors in V_ν^* invariant.

As representations of $\mathfrak{g}$, there is an isomorphism $V_\nu^* \simeq V_{-w_0(\nu)} = V_{\nu^*}$ which is unique up to a scalar. It intertwines the action of σ^* on V_ν^* and the action of σ on V_{ν^*}. Note that there is a natural isomorphism $\mathrm{Hom}_{\mathfrak{g}}(V_\nu, V_{\vec{\lambda}}) \simeq (V_{\vec{\lambda}} \otimes V_\nu^*)^{\mathfrak{g}}$, which is σ-equivariant. This concludes the proof. $\square$

The following theorem was proved in [13]. We give a simple proof here using Jantzen formula directly.

Theorem 5.6 *([13]). Let $\vec{\lambda}$ be a tuple of dominant weights of $\mathfrak{g}$. We have $\mathrm{tr}(\sigma|V_{\vec{\lambda}}^{\mathfrak{g}}) = \dim W_{\vec{\lambda}}^{\mathfrak{g}_\sigma}$.*

Proof. On one hand, from the decomposition $V_{\vec{\lambda}} \simeq \bigoplus_{\nu \in P^+} \mathrm{Hom}_{\mathfrak{g}}(V_\mu, V_{\vec{\lambda}}) \otimes V_\mu$, we have

$$\mathrm{ch}_\sigma(V_{\vec{\lambda}}) = \sum_{\mu \in (P^+)^\sigma} \mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_\mu, V_{\vec{\lambda}}))\mathrm{ch}_\sigma(V_\mu).$$

On the other hand, we have the following equalities

$$\begin{aligned}\mathrm{ch}_\sigma(V_{\vec{\lambda}}) &= \mathrm{ch}_\sigma(V_{\lambda_1}) \cdots \mathrm{ch}_\sigma(V_{\lambda_k}) = \mathrm{ch}(W_{\lambda_1}) \cdots \mathrm{ch}(W_{\lambda_k}) \\ &= \mathrm{ch}(W_{\vec{\lambda}}) = \sum \dim \mathrm{Hom}_{\mathfrak{g}_\sigma}(W_\mu, W_{\vec{\lambda}})\mathrm{ch}(W_\mu),\end{aligned}$$

where the first equality follows from Lemma 5.4 and the second equality follows from Theorem 5.3. In view of Lemma 5.5, the theorem follows. $\square$

Let $R(\mathfrak{g}_\sigma)$ denote the representation ring of $\mathfrak{g}_\sigma$.

Proposition 5.7. *There is a natural ring isomorphism $R(\mathfrak{g},\sigma)\simeq R(\mathfrak{g}_\sigma)$ by sending $[V_\lambda]_\sigma\mapsto [W_\lambda]$ for any σ-invariant dominant weight λ.*

Proof. For any $\lambda,\mu\in (P^+)^\sigma$, consider the following two decompositions

$$[V_\lambda]_\sigma\otimes [V_\mu]_\sigma=\sum_{\sigma(\mu)=\mu}\mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_\nu,V_\lambda\otimes V_\mu))[V_\nu]_\sigma,$$

$$[W_\lambda]\otimes [W_\mu]=\sum_{\sigma(\mu)=\mu}\dim \mathrm{Hom}_{\mathfrak{g}}(W_\nu,W_\lambda\otimes W_\mu)[W_\nu].$$

In view of Theorem 5.6 and Lemma 5.5, we have

$$\mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_\nu,V_\lambda\otimes V_\mu))=\dim \mathrm{Hom}_{\mathfrak{g}_\sigma}(W_\nu,W_\lambda\otimes W_\mu).$$

Hence the proposition follows. □

5.2. A new definition of σ-twisted fusion ring via Borel-Weil-Bott theory

Lemma 5.8. *The operation $[\cdot]_\sigma$ satisfies Euler-Poincaré property, i.e. for any complex of finite dimensional $\mathfrak{g}\rtimes\langle\sigma\rangle$-representations*

$$V^\bullet := \cdots \xrightarrow{d_{i-1}} V^i \xrightarrow{d_i} V^{i+1} \xrightarrow{d_{i+1}} \cdots$$

such that only finite many V^i are nonzero, we have

$$[V^\bullet]_\sigma=\sum_i(-1)^i[H^i(V^\bullet)]_\sigma,$$

where $[V^\bullet]_\sigma:=\sum_i(-1)^i[V^i]_\sigma$, and $H^i(V^\bullet)$ is the i-th cohomology of this complex.

Proof. First of all, we have Euler-Poincaré property in the representation ring $R(\mathfrak{g}\rtimes\langle\sigma\rangle)$ of $\mathfrak{g}\rtimes\langle\sigma\rangle$, i.e.

$$\sum_i(-1)^i[V^i]=\sum_i(-1)^i[H^i(V^\bullet)].$$

Secondly we can define a linear map $R(\mathfrak{g}\rtimes\langle\sigma\rangle)\to R(\mathfrak{g},\sigma)$ given by $[V]\mapsto [V]_\sigma$. It is well-defined and additive, since any finite dimensional representation of $\mathfrak{g}\rtimes\langle\sigma\rangle$ is completely reducible. Hence the lemma follows. □

Recall the σ-twisted fusion ring $R_\ell(\mathfrak{g},\sigma)$ defined in Section 3.5. We embed $R_\ell(\mathfrak{g},\sigma)$ into $R(\mathfrak{g},\sigma)$ as free abelian groups by simply sending λ to $[V_\lambda]_\sigma$ for any $\lambda\in P^\sigma_\ell$. From now on we view $R_\ell(\mathfrak{g},\sigma)$ as a free abelian group with basis $\{[V_\lambda]_\sigma\,|\,\lambda\in P^\sigma_\ell\}$. The fusion product $\lambda\cdot\mu$ in $R_\ell(\mathfrak{g},\sigma)$ will be written as $[V_\lambda]_\sigma\cdot[V_\mu]_\sigma$.

Given any integrable representation $\mathcal{H}$ of $\hat{\mathfrak{g}}$, we denote by $\mathcal{H}_{\hat{\mathfrak{g}}^-}$ the coinvariant space of $\hat{\mathfrak{g}}^-$ on $\mathcal{H}$. If $\mathcal{H}$ is a representation of $\hat{\mathfrak{g}} \rtimes \langle\sigma\rangle$, then the space $\mathcal{H}_{\hat{\mathfrak{g}}^-}$ is naturally a representation of $\mathfrak{g} \rtimes \langle\sigma\rangle$. For any $\lambda, \mu \in P_\ell^\sigma$, we define

$$[V_\lambda]_\sigma \otimes_\ell [V_\mu]_\sigma := [(H^*(\mathrm{Gr}_G, \mathcal{L}_\ell(V_\lambda \otimes V_\mu))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma \in R_\ell(\mathfrak{g}, \sigma), \tag{42}$$

where we view $(H^*(\mathrm{Gr}_G, \mathcal{L}_\ell(V_\lambda \otimes V_\mu))^\vee)_{\hat{\mathfrak{g}}^-}$ as a complex of $\mathfrak{g} \rtimes \langle\sigma\rangle$-representations with zero differentials.

Note that all representations of $\hat{\mathfrak{g}}$ appearing in $H^*(\mathrm{Gr}_G, \mathcal{L}_\ell(V_\lambda \otimes V_\mu))^\vee$ are of level ℓ, and only finite many cohomology groups are nonzero. Hence the above definition makes sense.

Recall the representation $\mathcal{H}_\nu \otimes V_\mu^z$ defined in Section 4.3. The following is a vanishing theorem of Lie algebra cohomology due to Teleman [32].

Theorem 5.9. *For any $\lambda, \mu, \nu \in P_\ell$ and for any $i \geq 1$, V_λ does not occur in $H_i(\hat{\mathfrak{g}}^-, \mathcal{H}_\nu \otimes V_\mu^z)$ as a $\mathfrak{g}$-representation.*

We now show that the product defined in (42) is exactly the fusion product.

Theorem 5.10. *Two products on $R_\ell(\mathfrak{g}, \sigma)$ coincide, i.e. for any $\lambda, \mu \in P_\ell^\sigma$, we have $[V_\lambda]_\sigma \otimes_\ell [V_\mu]_\sigma = [V_\lambda]_\sigma \cdot [V_\mu]_\sigma$.*

Proof. Consider the following decomposition

$$V_\lambda \otimes V_\mu = \bigoplus_\nu \mathrm{Hom}_{\mathfrak{g}}(V_\nu, V_\lambda \otimes V_\mu) \otimes V_\nu.$$

By the fact (32), we may in further write

$$V_\lambda \otimes V_\mu \simeq \bigoplus_{w \in W^\dagger_{\ell+\check{h}}, \nu \in P_\ell} \mathrm{Hom}_{\mathfrak{g}}(V_{w\star\nu}, V_\lambda \otimes V_\mu) \otimes V_{w\star\nu}. \tag{43}$$

We have the following chain of equalities

$$\begin{aligned}
[V_\lambda]_\sigma \otimes_\ell [V_\mu]_\sigma &= \sum_i (-1)^i [(H^i(\mathrm{Gr}_G, \mathcal{L}_\ell(V_\lambda \otimes V_\mu))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma \\
&= \sum_i (-1)^i \sum_{\substack{w \in (W^\dagger_{\ell+\check{h}})^\sigma \\ \ell(w)=i, \nu \in P_\ell^\sigma}} [\mathrm{Hom}_{\mathfrak{g}}(V_{w\star\nu}, V_\lambda \otimes V_\mu) \otimes (H^i(\mathrm{Gr}_G, \mathcal{L}_\ell(V_{w\star\nu}))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma \\
&= \sum_{w \in (W^\dagger_{\ell+\check{h}})^\sigma, \nu \in P_\ell^\sigma} (-1)^{\ell(w)} [\mathrm{Hom}_{\mathfrak{g}}(V_{w\star\nu}, V_\lambda \otimes V_\mu) \otimes V_{\nu, \epsilon_w}]_\sigma \\
&= \sum_{w \in (W^\dagger_{\ell+\check{h}})^\sigma, \nu \in P_\ell^\sigma} (-1)^{\ell_\sigma(w)} \mathrm{tr}(\sigma | \mathrm{Hom}_{\mathfrak{g}}(V_{w\star\nu}, V_\lambda \otimes V_\mu)) [V_\nu]_\sigma,
\end{aligned}$$

where the second equality follows from the decomposition (43), the third equality follows from Corollary 4.14 and the fourth equality follows from Lemma 5.1. By Lemma 3.7 and Proposition 3.8, we have the following σ-equivariant isomorphisms:

$$\begin{aligned} V_{\mathfrak{g},\ell,\lambda,\mu,\nu^*}(\mathbb{P}^1,0,1,\infty) &\simeq V_{\mathfrak{g},\ell,\lambda^*,\mu^*,\nu}(\mathbb{P}^1,0,1,\infty) \\ &\simeq (\mathcal{H}_\nu \otimes V_{\lambda^*}^\infty \otimes V_{\mu^*}^1)_{\mathfrak{g}[t^{-1}]} \\ &\simeq \mathrm{Hom}_{\mathfrak{g}}(V_\lambda, H_0(\hat{\mathfrak{g}}^-, \mathcal{H}_\nu \otimes V_{\mu^*}^1)). \end{aligned}$$

The following formula follows immediately from Theorem 5.9

$$\mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_\lambda, H_0(\hat{\mathfrak{g}}^-, \mathcal{H}_\nu \otimes V_{\mu^*}^1))) = \sum_i (-1)^i \mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_\lambda, H_i(\hat{\mathfrak{g}}^-, \mathcal{H}_\nu \otimes V_{\mu^*}^1))).$$

By Lemma 5.8 and Theorem 4.16, we have

$$\begin{aligned} &\sum (-1)^i \mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_\lambda, H_i(\hat{\mathfrak{g}}^-, \mathcal{H}_\nu \otimes V_\mu^1))) \\ &= \sum_i (-1)^i \sum_{w\in (W_{\ell+\check{h}}^\dagger)^\sigma, \ell(w)=i} \mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V_{w\star\nu,\epsilon_w} \otimes V_{\mu^*})) \\ &= \sum_{w\in (W_{\ell+\check{h}}^\dagger)^\sigma} (-1)^{\ell_\sigma(w)} \mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V_{w\star\nu} \otimes V_{\mu^*})). \end{aligned}$$

It follows that

$$\begin{aligned} [V_\lambda]_\sigma \cdot [V_\mu]_\sigma &= \sum_{\nu\in P_\ell^\sigma} \mathrm{tr}(\sigma|V_{\mathfrak{g},\ell,\lambda,\mu,\nu^*}(\mathbb{P}^1,0,1,\infty))[V_\nu]_\sigma \\ &= \sum_{w\in (W_{\ell+\check{h}}^\dagger)^\sigma, \nu\in P_\ell^\sigma} (-1)^{\ell_\sigma(w)} \mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V_{w\star\nu} \otimes V_{\mu^*}))[V_\nu]_\sigma. \end{aligned}$$

In the end, we need to check that

$$\mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_{w\star\nu}, V_\lambda \otimes V_\mu)) = \mathrm{tr}(\sigma|\mathrm{Hom}_{\mathfrak{g}}(V_\lambda, V_{w\star\nu} \otimes V_{\mu^*})).$$

In view of Lemma 5.5, it reduces to show that the trace of σ on $V^{\mathfrak{g}}_{w\star\nu^*,\lambda,\mu}$ and $V^{\mathfrak{g}}_{\lambda^*,w\star\nu,\mu^*}$ are equal. This is a consequence of Lemma 3.6. □

From the proof of Theorem 5.10, we get the following twisted analogue of Kac-Walton formula (in the usual setting, see [19,36]).

Theorem 5.11. *For any $\lambda,\mu,\nu \in P_\ell^\sigma$, we have*

$$\mathrm{tr}(\sigma|V_{\mathfrak{g},\lambda,\mu,\nu}(\mathbb{P}^1,0,1,\infty)) = \sum_{w\in (W_{\ell+\check{h}}^\dagger)^\sigma} (-1)^{\ell_\sigma(w)} \mathrm{tr}(\sigma|V^{\mathfrak{g}}_{\lambda,\mu,w\star\nu}).$$

Remark 5.12. The proofs of Theorem 5.10, 5.11 do not rely on the fact that the trace on conformal blocks is a fusion rule. In fact Theorem 5.11 is used to show that the trace on conformal blocks gives a fusion rule, see Lemma 3.16.

5.3. Ring homomorphism from σ-twisted representation ring to σ-twisted fusion ring

We first construct a $\mathbb{Z}$-linear map

$$\pi_\sigma : R(\mathfrak{g}, \sigma) \to R_\ell(\mathfrak{g}, \sigma).$$

For any finite dimensional $\mathfrak{g} \rtimes \langle\sigma\rangle$-representation V, we define

$$\pi_\sigma([V]_\sigma) := [(H^*(\mathrm{Gr}_G, \mathcal{L}_\ell(V))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma \in R_\ell(\mathfrak{g}, \sigma).$$

Lemma 5.13. *For any $w \in (W^\dagger_{\ell+\check{h}})^\sigma$ and $\lambda \in (P^+)^\sigma$, we have*

$$[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(w \star \lambda))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma = (-1)^{\ell_\sigma(w)}[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(\lambda))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma.$$

Proof. We can write $\lambda = y\star\lambda_0$ where $y \in (W^\dagger_{\ell+\check{h}})^\sigma$ and $\lambda_0 \in (P_\ell)^\sigma$. Then $w\star\lambda = (wy)\star\lambda_0$. In view of Theorem 4.1 and Theorem 4.10, we have

$$\begin{aligned}[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(w \star \lambda))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma &= (-1)^{\ell_\sigma(wy)}[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(\lambda_0))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma \\ &= (-1)^{\ell_\sigma(w)}[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(\lambda))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma.\end{aligned}$$

Hence the lemma follows. □

Proposition 5.14. *Given a finite dimensional representation V of $\mathfrak{g}\rtimes\langle\sigma\rangle$. For any $\lambda \in P^\sigma_\ell$ and $w \in (W^+_{\ell+\check{h}})^\sigma$, the following equality holds in $R_\ell(\mathfrak{g}, \sigma)$*

$$[(H^*(\mathrm{Gr}_G, \mathcal{L}_\ell(V_{w\star\lambda} \otimes V))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma = (-1)^{\ell_\sigma(w)}[(H^*(\mathrm{Gr}_G, \mathcal{L}_\ell(V_\lambda \otimes V))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma.$$

Proof. In view of Lemma 4.12, it suffices to show that

$$[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(\mathbb{C}_{w\star\lambda} \otimes V|_{B\rtimes\langle\sigma\rangle}))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma = (-1)^{\ell_\sigma(w)}[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(\mathbb{C}_\lambda \otimes V|_{B\rtimes\langle\sigma\rangle}))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma.$$

Note that there exists a filtration of $B \rtimes \langle\sigma\rangle$-representations

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = V$$

on V, such that for each i,

$$V_i/V_{i-1} \simeq \begin{cases} V(\mu) & \text{if } \sigma(\mu) = \mu \\ \bigoplus_{i=0}^{r-1} V(\sigma^i(\mu)) & \text{otherwise} \end{cases},$$

where $V(\mu)$ denotes the μ-weight space of V. By Lemma 5.1, it is easy to check that

$$
\begin{aligned}
&[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(\mathbb{C}_\lambda \otimes \bigoplus_{i=0}^{r-1} V(\sigma^i(\mu)))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma \\
&= \begin{cases} \mathrm{tr}(\sigma|V(\mu))[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(\lambda+\mu))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma & \text{if } \sigma(\mu) = \mu \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}
$$

Hence we get the following isomorphisms

$$
\begin{aligned}
[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(\mathbb{C}_\lambda \otimes V|_{B\rtimes\langle\sigma\rangle}))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma &= \sum_i [(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(\mathbb{C}_\lambda \otimes V_i/V_{i-1}))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma \\
&= \sum_{\mu\in P^\sigma} \mathrm{tr}(\sigma|V(\mu))[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(\lambda+\mu))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma.
\end{aligned}
$$

Similarly, we have

$$
[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(\mathbb{C}_{w\star\lambda} \otimes V|_{B\rtimes\langle\sigma\rangle}))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma = \sum_{\mu\in P^\sigma} \mathrm{tr}(\sigma|V(\mu))[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(w\star\lambda+\mu))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma.
$$

We can write w as $w = \tau_\beta y^{-1}$, where $y \in W^\sigma$ and τ_β is the translation for $\beta \in (\ell+\check{h})Q^\sigma$. It is easy to check that $w \star \lambda + \mu = w \star (\lambda + y \cdot \mu)$.

Since V is a representation of $\mathfrak{g} \rtimes \langle\sigma\rangle$, for any $y \in W^\sigma$ we have $\mathrm{tr}(\sigma|V(\mu)) = \mathrm{tr}(\sigma|V(y\cdot\mu))$, where $V(\mu)$ and $V(y\cdot\mu)$ denote the weight spaces of V as representation of $\mathfrak{g}$. We have the following chain of equalities

$$
\begin{aligned}
&[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(\mathbb{C}_{w\star\lambda} \otimes V|_{B\rtimes\langle\sigma\rangle}))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma \\
&= \sum_{\mu\in P^\sigma} \mathrm{tr}(\sigma|V(\mu))[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(w\star(\lambda+\mu)))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma \\
&= \sum_{\mu\in P^\sigma} \mathrm{tr}(\sigma|V(\mu))(-1)^{\ell_\sigma(w)}[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(\lambda+\mu))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma \\
&= (-1)^{\ell_\sigma(w)}[(H^*(\mathrm{Fl}_G, \mathcal{L}_\ell(\mathbb{C}_\lambda \otimes V|_{B\rtimes\langle\sigma\rangle}))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma,
\end{aligned}
$$

where the second isomorphism follows from Lemma 5.13. This finishes the proof. □

Proposition 5.15. *If $\lambda \in (P^+)^\sigma$ and $\lambda+\rho$ is in an affine wall of $W_{\ell+\check{h}}$, then*

$$
[(H^*(\mathrm{Gr}_G, \mathcal{L}_\ell(V_\lambda \otimes V))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma = 0.
$$

Proof. By Part (3) of Proposition 2.7, $\lambda+\rho$ is in an affine wall of $W^\sigma_{\ell+\check{h}}$, where by (13), $W^\sigma_{\ell+\check{h}} \simeq W^\sigma \ltimes (\ell+\check{h})\iota(Q^\sigma)$. Hence in view of Lemma 2.3, we can assume that $\lambda+\rho$ is in the following affine wall of $W^\sigma_{\ell+\check{h}}$ in $P^\sigma \otimes \mathbb{R}$,

$$H_{\alpha_\sigma,a} = \{\lambda+\rho \in P^\sigma \otimes \mathbb{R} \mid \langle \lambda+\rho, \check{\alpha}_\sigma\rangle = a\},$$

where $\check{\alpha}_\sigma$ is the coroot of a root α_σ of $\mathfrak{g}_\sigma$, and

$$a \in \begin{cases} (\ell+\check{h})\mathbb{Z} & \text{if } \mathfrak{g} \text{ is not of type } A_{2n} \\ \frac{\ell+\check{h}}{2}\mathbb{Z} & \text{if } \mathfrak{g} = A_{2n} \end{cases}.$$

Equivalently,

$$(\tau_{a\alpha_\sigma} \cdot s_{\alpha_\sigma}) \star (\lambda) = (s_{\alpha_\sigma} \cdot \tau_{-a\alpha_\sigma}) \star (\lambda) = \lambda, \tag{44}$$

where s_{α_σ} is the reflection with respect to α_σ in $W^\sigma_{\ell+\check{h}}$ and $\tau_{a\alpha_\sigma}$ is the translation by $a\alpha_\sigma$. Moreover,

$$(-1)^{\ell_\sigma(\tau_{a\alpha_\sigma}\cdot s_{\alpha_\sigma})} = (-1)^{\ell_\sigma(\tau_{a\alpha_\sigma})}(-1)^{\ell_\sigma(s_{\alpha_\sigma})} = -1,$$

since by Lemma 2.8, $\ell_\sigma(\tau_{a\alpha_\sigma})$ is an even integer.

By Proposition 5.14 we have

$$[(H^*(\mathrm{Gr}_G, \mathcal{L}_\ell(V_\lambda \otimes V))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma = -[(H^*(\mathrm{Gr}_G, \mathcal{L}_\ell(V_\lambda \otimes V))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma.$$

Hence $[(H^*(\mathrm{Gr}_G, \mathcal{L}_\ell(V_\lambda \otimes V))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma = 0$. □

Theorem 5.16. *The linear map $\pi_\sigma : R(\mathfrak{g},\sigma) \to R_\ell(\mathfrak{g},\sigma)$ is a ring homomorphism.*

Proof. By Theorem 5.10, we can use the product $\otimes_\ell$ for $R_\ell(\mathfrak{g},\sigma)$. We need to check that for any $\lambda,\mu \in (P^+)^\sigma$,

$$\pi_\sigma([V_\lambda \otimes V_\mu]_\sigma) = \pi_\sigma([V_\lambda]_\sigma) \otimes_\ell \pi_\sigma([V_\mu]_\sigma). \tag{45}$$

If $\lambda+\rho$ or $\mu+\rho$ is in an affine Wall, then by Proposition 5.15, both sides of (45) are zero. Hence (45) holds.

If $\lambda+\rho$ and $\mu+\rho$ are not in any affine Wall, let $\lambda_0 \in P^\sigma_\ell$ such that $w_\lambda \star \lambda_0 = \lambda$ and let $\mu_0 \in P^\sigma_\ell$ such that $w_\mu \star \mu_0 = \mu$ where $w_\lambda, w_\mu \in (W^\dagger_{\ell+\check{h}})^\sigma$, then

$$\begin{aligned}
\pi_\sigma([V_\lambda \otimes V_\mu]_\sigma) &= [(H^*(\mathrm{Gr}_G, \mathcal{L}_\ell(V_\lambda \otimes V_\mu))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma \\
&= (-1)^{\ell_\sigma(w_\lambda)}[(H^*(\mathrm{Gr}_G, \mathcal{L}_\ell(V_{\lambda_0} \otimes V_\mu))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma \\
&= (-1)^{\ell_\sigma(w_\lambda)+\ell_\sigma(w_\mu)}[(H^*(\mathrm{Gr}_G, \mathcal{L}_\ell(V_{\lambda_0} \otimes V_{\mu_0}))^\vee)_{\hat{\mathfrak{g}}^-}]_\sigma \\
&= (-1)^{\ell_\sigma(w_\lambda)+\ell_\sigma(w_\mu)}[V_{\lambda_0}]_\sigma \otimes_\ell [V_{\mu_0}]_\sigma \\
&= \pi_\sigma([V_\lambda]_\sigma) \otimes_\ell \pi_\sigma([V_\mu]_\sigma),
\end{aligned}$$

where the second, the third and the fifth equalities follows from Proposition 5.14, and the fourth equality is the definition (42). This finishes the proof of the theorem. □

We can explicitly describe the map π_σ.

Corollary 5.17. *The map $\pi_\sigma : R(\mathfrak{g},\sigma) \to R_\ell(\mathfrak{g},\sigma)$ can be described as follows, for any $\lambda \in (P^+)^\sigma$ we have*

$$\pi_\sigma([V_\lambda]_\sigma) = \begin{cases} 0 & \text{if } \lambda+\rho \text{ belongs to an affine Wall of } W_{\ell+\check{h}} \text{ in } P_{\mathbb{R}} \\ (-1)^{\ell_\sigma(w)}[V_{w^{-1}\star\lambda}]_\sigma & \text{if } w^{-1}\star\lambda \in P_\ell^\sigma \text{ for some } w \in (W^\dagger_{\ell+\check{h}})^\sigma. \end{cases}$$

Proof. The corollary is an immediate consequence of Corollary 4.14, Proposition 5.14 and Proposition 5.15. □

5.4. Characters of the σ-twisted fusion ring

In Section 5.4 and Section 5.5 we basically follow the arguments in [1, Section 9]. However our arguments of Lemma 5.21 and Proposition 5.23 are substantially different, since in our setting there is no natural identification between $P_\sigma/(\ell+\check{h})\iota(Q^\sigma)$ and $T_{\sigma,\ell}$.

Recall that P_σ (resp. Q_σ) is the weight lattice (resp. root lattice) of $\mathfrak{g}_\sigma$, and the bijection map $\iota : P_\sigma \simeq P^\sigma$ defined in Section 2.1.

Let $\mathbb{Z}[P_\sigma]$ be the group ring of P_σ; we denote by $(e^\lambda)_{\lambda\in P_\sigma}$ its basis so that the multiplication in $\mathbb{Z}[P_\sigma]$ obeys the rule $e^\lambda e^\mu = e^{\lambda+\mu}$. The action of W_σ and $W^\sigma_{\ell+\check{h}} \simeq W_\sigma \ltimes (\ell+\check{h})\iota(Q^\sigma)$ on P_σ extends to $\mathbb{Z}[P_\sigma]$. We denote by $\mathbb{Z}[P_\sigma]_{W^\sigma}$ (resp. $\mathbb{Z}[P_\sigma]_{W^\sigma_{\ell+\check{h}}}$) the quotient of $\mathbb{Z}[P_\sigma]$ by the sublattice spanned by $e^\lambda - (-1)^{\ell_\sigma(w)}e^{w\star\lambda}$ for any $w \in W_\sigma$ (resp. $w \in W^\sigma_{\ell+\check{h}}$). Let $p : \mathbb{Z}[P_\sigma]_{W_\sigma} \to \mathbb{Z}[P_\sigma]_{W^\sigma_{\ell+\check{h}}}$ be the projection map.

Lemma 5.18. *The kernel $\ker(p)$ is spanned by the class of $e^{\lambda+\alpha} - e^\lambda$ in $\mathbb{Z}[P_\sigma]_{W_\sigma}$, for $\lambda \in P_\sigma$ and $\alpha \in (\ell+\check{h})\iota(Q^\sigma)$.*

Proof. We first define a group action $\bullet$ of $W^\sigma_{\ell+\check{h}}$ on $\mathbb{Z}[P_\sigma]$. For any $e^\lambda \in \mathbb{Z}[P_\sigma]$ and $w\tau_\alpha \in W^\sigma_{\ell+\check{h}}$ where $w \in W_\sigma$ and $\alpha \in (\ell+\check{h})\iota(Q^\sigma)$, define

$$w\tau_\alpha \bullet e^\lambda := (-1)^{\ell_\sigma(w\tau_\alpha)}e^{w\star(\lambda+\alpha)}.$$

It is easy to see that this gives a group action of $W^\sigma_{\ell+\check{h}}$ on $\mathbb{Z}[P_\sigma]$. Note that in the above formula, $(-1)^{\ell_\sigma(w\tau_\alpha)} = (-1)^{\ell_\sigma(w)}$, since by Lemma 2.8, $\ell_\sigma(\tau_\alpha)$ is even.

Let $\mathbb{Z}[P_\sigma]_{(\ell+\check{h})\iota(Q^\sigma)}$ denote the space of coinvariants of $\mathbb{Z}[P_\sigma]$ with respect to the translation action of $(\ell+\check{h})\iota(Q^\sigma)$. Consider the following short exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{Z}[P_\sigma] \longrightarrow \mathbb{Z}[P_\sigma]_{(\ell+\check{h})\iota(Q^\sigma)} \longrightarrow 0\ ,$$

where K is the sublattice of $\mathbb{Z}[P_\sigma]$ spanned by $e^{\lambda+\alpha} - e^\lambda$ for $\lambda \in P_\sigma$ and $\alpha \in (\ell+\check{h})\iota(Q^\sigma)$. With respect to the action $\bullet$ of W_σ, we apply the functor of W_σ-coinvariants to the above short exact sequence. Since coinvariant functor is right exact, we get the following exact sequence

$$K_{W_\sigma} \longrightarrow \mathbb{Z}[P_\sigma]_{W_\sigma} \longrightarrow (\mathbb{Z}[P_\sigma]_{(\ell+\check{h})\iota(Q^\sigma)})_{W_\sigma} \longrightarrow 0\ .$$

Observe that

$$\mathbb{Z}[P_\sigma]_{W^\sigma_{\ell+\check{h}}} = (\mathbb{Z}[P_\sigma]_{(\ell+\check{h})\iota(Q^\sigma)})_{W_\sigma}.$$

This concludes the proof of the lemma. □

By Proposition 5.7 and Theorem 5.16 we get a ring homomorphism $\tilde{\pi}_\sigma : R(\mathfrak{g}_\sigma) \simeq R(\mathfrak{g},\sigma) \to R_\ell(\mathfrak{g},\sigma)$. Let ϕ_σ be the map $R(\mathfrak{g}_\sigma) \to \mathbb{Z}[P_\sigma]_{W_\sigma}$ sending $[W_\lambda]$ to the class of e^λ. Similarly, let $\phi_{\sigma,\ell}$ be the map $R_\ell(\mathfrak{g},\sigma) \to \mathbb{Z}[P_\sigma]_{W^\sigma_{\ell+\check{h}}}$ sending $[V_\lambda]_\sigma$ to the class e^λ for any $\lambda \in P^\sigma_\ell$. By the same arguments as in [1, Section 8], ϕ_σ and $\phi_{\sigma,\ell}$ are bijections. As a consequence of Corollary 5.17, the following diagram commutes

$$\begin{tikzcd} R(\mathfrak{g}_\sigma) \arrow[r, "\tilde{\pi}_\sigma"] \arrow[d, "\phi_\sigma"] & R_\ell(\mathfrak{g},\sigma) \arrow[d, "\phi_{\sigma,\ell}"] \\ \mathbb{Z}[P_\sigma]_{W_\sigma} \arrow[r, "p"] & \mathbb{Z}[P_\sigma]_{W^\sigma_{\ell+\check{h}}} \end{tikzcd} \quad . \tag{46}$$

For any $\lambda \in P_\sigma$, put

$$J(e^{\lambda+\rho}) = \sum_{w\in W_\sigma} (-1)^{\ell_\sigma(w)} e^{w(\lambda+\rho_\sigma)}, \tag{47}$$

where ρ_σ is the sum of all fundamental weights of $\mathfrak{g}_\sigma$. Recall that $\iota(\rho) = \rho_\sigma$ via the bijection $\iota : P^\sigma \simeq P_\sigma$. By Weyl character formula, for any $\lambda \in P^+_\sigma$ and $t \in T_\sigma$, we have $\mathrm{tr}(t|W_\lambda) = \frac{J(e^{\lambda+\rho_\sigma})(t)}{J(e^{\rho_\sigma})(t)}$. Let $T_{\sigma,\ell}$ be the finite subgroup of T_σ given by

$$T_{\sigma,\ell} := \{t \in T_\sigma \mid e^\alpha(t) = 1, \alpha \in (\ell+\check{h})\iota(Q^\sigma)\}.$$

Proposition 5.19. *For any $t \in T_{\sigma,\ell}$, the character $\mathrm{tr}(t|\cdot)$ factors through $\tilde{\pi}_\sigma : R(\mathfrak{g}_\sigma) \to R_\ell(\mathfrak{g},\sigma)$.*

Proof. Let $j_t : \mathbb{Z}[P_\sigma]_{W^\sigma} \to \mathbb{C}$ be the additive map such that for any $\lambda \in P_\sigma$, $j_t(e^\lambda) = \frac{J(e^{\lambda+\rho_\sigma})(t)}{J(e^{\rho_\sigma})(t)}$. By the definition of $\mathbb{Z}[P_\sigma]_{W^\sigma}$ and $J(\cdot)$, it is easy to check that j_t is well-defined. By Weyl character formula, the following diagram commutes:

$$\begin{tikzcd} R(\mathfrak{g}_\sigma) \arrow[r, "\phi_\sigma"] \arrow[dr, "\mathrm{tr}(t|\cdot)"'] & \mathbb{Z}[P_\sigma]_{W_\sigma} \arrow[d, "j_t"] \\ & \mathbb{C} \end{tikzcd} .$$

By the commutativity of the diagram (46) and Lemma 5.18, to show $\mathrm{tr}(t|\cdot)$ factors through $\tilde{\pi}_\sigma$, we need to check that j_t takes the value zero on $e^{\lambda+\alpha} - e^{\lambda}$ for any $\lambda \in P_\sigma$ and $\alpha \in (\ell + \check{h})\iota(Q^\sigma)$. Since t satisfies that $e^\alpha(t) = 0$ for any $\alpha \in (\ell + \check{h})\iota(Q^\sigma)$, it is clear that j_t takes the value zero on $e^{\lambda+\alpha} - e^{\lambda}$. This concludes the proof. $\square$

An element $t \in T_\sigma$ is regular if the stabilizer of W_σ at t is trivial. We denote by $T^{\mathrm{reg}}_{\sigma,\ell}$ the set of regular elements in $T_{\sigma,\ell}$. Let $\check{\rho}_\sigma$ denotes the sum of all fundamental coweights of $\mathfrak{g}_\sigma$. Consider the short exact sequence

$$0 \to 2\pi i \check{Q}_\sigma \to \mathfrak{t}_\sigma \to T_\sigma \to 1,$$

where $\check{Q}_\sigma$ denote the dual root lattice of $\mathfrak{g}_\sigma$ and $\mathfrak{t}_\sigma$ denotes the Cartan subalgebra of $\mathfrak{g}_\sigma$. Let $\check{L}_\sigma$ be the dual lattice of $\iota(Q^\sigma)$ in $\mathfrak{t}_\sigma$. We have the following natural isomorphism

$$T_{\sigma,\ell} \simeq (\frac{1}{\ell + \check{h}} \check{L}_\sigma)/\check{Q}_\sigma \simeq \check{L}_\sigma/(\ell + \check{h})\check{Q}_\sigma. \tag{48}$$

For any $\check{\mu} \in \check{L}_\sigma$, we denote by $t_{\check{\mu}}$ the associated element of $\check{\mu} + \check{\rho}_\sigma$ in $T_{\sigma,\ell}$.

We put $\check{P}_{\sigma,\ell} := \{\check{\mu} \in \check{P}^+_\sigma \mid \langle \check{\mu}, \theta_\sigma \rangle_\sigma \leq \ell\}$, where θ_σ denotes the highest root of $\mathfrak{g}_\sigma$ and $\check{P}^+_\sigma$ denotes the set of dominant coweights of $\mathfrak{g}_\sigma$.

Lemma 5.20. *Assume that $\mathfrak{g} \neq A_{2n}$. There exists a bijection $\check{P}_{\sigma,\ell} \simeq T^{\mathrm{reg}}_{\sigma,\ell}/W_\sigma$ with the map given by $\check{\mu} \mapsto t_{\check{\mu}}$,*

Proof. When $\mathfrak{g} \neq A_{2n}$, by Lemma 2.3 $\iota(Q^\sigma) = Q_\sigma$. Thus $\check{L}_\sigma = \check{P}_\sigma$. We observe that $\langle \check{\rho}_\sigma, \theta_\sigma \rangle = \check{h} - 1$ where $\check{h}$ is the dual Coxeter number of $\mathfrak{g}$. This can be read from [14, Table 2, p. 66]). It follows that

$$\check{P}_{\sigma,\ell} = \{\check{\mu} \in \check{P}^+_\sigma \mid \langle \check{\mu} + \check{\rho}_\sigma, \theta_\sigma \rangle_\sigma < \ell + \check{h}\},$$

i.e. $\check{P}_{\sigma,\ell}$ consists of all points of $\check{P}^+_\sigma$ sitting in the interior of the fundamental alcove with respect to the action of the affine Weyl group $W_\sigma \ltimes (\ell + \check{h})\check{Q}_\sigma$. From the isomorphism (48), we can see that any W_σ-orbit in $T^{\mathrm{reg}}_{\sigma,\ell}$ has a unique representative in $\check{P}_{\sigma,\ell}$. Hence the lemma follows. $\square$

Lemma 5.21. *The cardinality of $T^{\mathrm{reg}}_{\sigma,\ell}/W_\sigma$ is the equal to the cardinality of P^σ_ℓ.*

Proof. When $\mathfrak{g}$ is of type A_{2n}, by Lemma 2.3, $\iota(Q^\sigma) = \frac{1}{2}Q_{\sigma,\ell}$ where $Q_{\sigma,\ell}$ is the lattice spanned by long roots of G_σ. The proof of this lemma is exactly the same as the proof of [1, Lemma 9.3]. We omit the detail.

Now we assume $\mathfrak{g} \neq A_{2n}$. Put

$$P_{\sigma,\ell} := \{\lambda \in P_\sigma^+ \,|\, \langle \lambda, \check{\theta}_\sigma \rangle_\sigma \leq \ell\},$$

where $\check{\theta}_\sigma$ denotes the highest coroot of $\mathfrak{g}_\sigma$. In view of (8) and Lemma 2.1, the map ι induces a natural bijection $\iota : P_\ell^\sigma \simeq P_{\sigma,\ell}$.

In view of Lemma 5.20, we are reduced to show that $\check{P}_{\sigma,\ell}$ and $P_{\sigma,\ell}$ have the same cardinality. If $\mathfrak{g}_\sigma$ is not of type B_n or C_n, it is true, since in this case weight lattice and coweight lattice, root lattice and coroot lattice can be identified. Otherwise, if $\mathfrak{g}_\sigma$ is of type B_n or C_n, by comparing the highest roots of B_n and C_n (see [14, Table 2, p. 66]), we conclude that $\check{P}_{\sigma,\ell}$ and $P_{\sigma,\ell}$ indeed have the same cardinality. □

The following proposition completely describes all characters of $R_\ell(\mathfrak{g}, \sigma)$.

Proposition 5.22. $\{\mathrm{tr}(t|\cdot) \,|\, t \in T_{\sigma,\ell}^{\mathrm{reg}}/W_\sigma\}$ *gives a full set of characters of* $R_\ell(\mathfrak{g}, \sigma)$.

Proof. This is an immediate consequence of Proposition 5.19 and Lemma 5.21. □

5.5. Proof of Theorem 1.2

Let $\check{T}_{\sigma,\ell}$ denote the finite abelian subgroup $\check{T}_{\sigma,\ell} := P_\sigma/(\ell + \check{h})\iota(Q^\sigma)$. For any $\lambda \in P_\sigma$, we denote by $\check{t}_\lambda$ the element in $\check{T}_{\sigma,\ell}$ associated to $\lambda + \rho_\sigma$.

Recall that Φ_σ is the set of roots of $\mathfrak{g}_\sigma$. In the following lemma we determine $\chi(\omega_\sigma)$ for each $\chi = \mathrm{tr}(t|\cdot)$, where ω_σ is the Casimir element defined in (22).

Proposition 5.23. *For any* $t \in T_{\sigma,\ell}^{\mathrm{reg}}$, *we have* $\sum_{\lambda \in P_\ell^\sigma} |\mathrm{tr}(t|W_\lambda)|^2 = \frac{|T_{\sigma,\ell}|}{\Delta_\sigma(t)}$, *where* $\Delta_\sigma = \prod_{\alpha \in \Phi_\sigma}(e^\alpha - 1)$.

Proof. When $\mathfrak{g} = A_{2n}$, the proof of this lemma is identical to the proof of [1, Lemma 9.7]. We omit the detail.

Now we assume $\mathfrak{g} \neq A_{2n}$. In this case, we have

$$\check{T}_{\sigma,\ell} = P_\sigma/(\ell + \check{h})Q_\sigma, \quad \text{and} \quad T_{\sigma,\ell} \simeq \check{P}_\sigma/(\ell + \check{h})\check{Q}_\sigma.$$

For any $\lambda \in P_\sigma$ and $\check{\mu} \in \check{L}_\sigma = \check{P}_\sigma$, we have

$$J(e^{\lambda+\rho_\sigma})(t_{\check{\mu}}) = \sum_{w \in W_\sigma} (-1)^{\ell_\sigma(w)} e^{2\pi i \frac{\langle \lambda+\rho_\sigma, w(\check{\mu}+\check{\rho}_\sigma)\rangle_\sigma}{\ell+\check{h}}} = J(e^{\check{\mu}+\check{\rho}_\sigma})(\check{t}_\lambda),$$

where we put $J(e^{\check{\mu}+\check{\rho}_\sigma}) = \sum_{w \in W_\sigma}(-1)^{\ell_\sigma(w)} e^{w(\check{\mu}+\check{\rho}_\sigma)}$. By Weyl character formula, we have

$$\sum_{\lambda\in P_\ell^\sigma} |\mathrm{tr}(t_{\check{\mu}}|W_\lambda)|^2 = \frac{1}{\Delta_\sigma(t_{\check{\mu}})} \sum_{\lambda\in P_\ell^\sigma} |J(e^{\check{\mu}+\check{\rho}_\sigma})(\check{t}_\lambda)|^2.$$

We now introduce an inner product $(\cdot,\cdot)$ on the space $L^2(\check{T}_{\sigma,\ell})$ of functions on the finite abelian group $\check{T}_{\sigma,\ell}$,

$$(\phi,\psi) := \frac{1}{|\check{T}_\ell|} \sum_{\check{t}\in\check{T}_{\sigma,\ell}} \phi(\check{t})\overline{\psi(\check{t})}, \quad \text{for any functions } \phi,\psi \text{ on } \check{T}_{\sigma,\ell}.$$

The function $J(e^{\check{\mu}+\check{\rho}_\sigma})$ on $\check{T}_{\sigma,\ell}$ is W_σ-antisymmetric, i.e. $J(e^{w\cdot(\check{\mu}+\check{\rho}_\sigma)}) = (-1)^{\ell_\sigma(w)} J(e^{\check{\mu}+\check{\rho}_\sigma})$. It shows that if t is not regular, then for any $\check{t}\in\check{T}_{\sigma,\ell}$, $J(e^{\check{\mu}+\check{\rho}_\sigma})(\check{t}) = 0$. It follows that

$$\sum_{\lambda\in P_\ell^\sigma} |J(e^{\check{\mu}+\check{\rho}_\sigma})(\check{t}_\lambda)|^2 = \frac{|\check{T}_{\sigma,\ell}|}{|W_\sigma|} ||J(e^{\check{\mu}+\check{\rho}_\sigma})||,$$

where $||J(e^{\check{\mu}+\check{\rho}_\sigma})|| = \sqrt{(J(e^{\check{\mu}+\check{\rho}_\sigma}), J(e^{\check{\mu}+\check{\rho}_\sigma}))}$.

If t is regular, in view of Lemma 5.20 we can assume $t = t_{\check{\mu}}$ where $\check{\mu}\in\check{P}_{\sigma,\ell}$. Now we show that the restriction of $e^{w\cdot(\check{\mu}+\check{\rho}_\sigma)}$ on $\check{T}_{\sigma,\ell}$ are all distinct. For any two distinct elements $w,w'\in W_\sigma$, if $e^{w(\check{\mu}+\check{\rho}_\sigma)}$ and $e^{w'(\check{\mu}+\check{\rho}_\sigma)}$ are equal on $\check{T}_{\sigma,\ell}$, it means that the pairing $\langle w(\check{\mu}+\check{\rho}_\sigma) - w'(\check{\mu}+\check{\rho}_\sigma), \lambda\rangle_\sigma \in (\ell+\check{h})\mathbb{Z}$ for any $\lambda\in P_\sigma$. Equivalently, $w(\check{\mu}+\check{\rho}_\sigma) - w'(\check{\mu}+\check{\rho}_\sigma)\in(\ell+\check{h})\check{Q}_\sigma$. It is impossible as $\check{\mu}+\check{\rho}_\sigma$ is in the fundamental alcove of the affine Weyl group $W_\sigma \ltimes (\ell+\check{h})\check{Q}_\sigma$.

By the orthogonality relation for the characters of $\check{T}_{\sigma,\ell}$, we have $||J(e^{\check{\mu}+\check{\rho}_\sigma})|| = |W_\sigma|$. Hence,

$$\sum_{\lambda\in P_\ell^\sigma} |\mathrm{tr}(t_{\check{\mu}}|W_\lambda)|^2 = \frac{|\check{T}_{\sigma,\ell}|}{\Delta_\sigma(t_{\check{\mu}})}.$$

From the non-degeneracy of the pairing $\check{T}_{\sigma,\ell}\times T_{\sigma,\ell}\to\mathbb{C}^\times$ given by $(\check{t}_\lambda, t_{\check{\mu}}) \mapsto e^{2\pi i\frac{\langle\lambda+\rho_\sigma,\check{\mu}+\check{\rho}_\sigma\rangle_\sigma}{\ell+\check{h}}}$, we have $|T_{\sigma,\ell}| = |\check{T}_{\sigma,\ell}|$. This concludes the proof of the proposition. □

Finally Theorem 1.2 follows from Proposition 3.19, Proposition 5.23, and Proposition 5.22.

5.6. A corollary of Theorem 1.2

Let σ be a nontrivial diagram automorphism on $\mathfrak{g} = sl_{2n+1}$. Then the orbit Lie algebra $\mathfrak{g}_\sigma$ is isomorphic to sp_{2n}.

Theorem 5.24. *With the same setting as in Theorem 1.2. If ℓ is an odd positive integer, then we have the following formula*

$$\mathrm{tr}(\sigma|V_{sl_{2n+1},\ell,\vec{\lambda}}(C,\vec{p})) = \dim V_{sp_{2n},\frac{\ell-1}{2},\vec{\lambda}}(C,\vec{p}).$$

Proof. By assumption, $\langle \lambda_i, \check{\theta}\rangle \leq \ell$ for any λ_i. In view of (8) and Lemma 2.1, we have $\langle \iota(\lambda_i), \check{\theta}_{\sigma,s}\rangle_\sigma \leq \ell/2$, where $\check{\theta}_{\sigma,s}$ is the coroot of the highest root θ_σ of $\mathfrak{g}_\sigma$. Since ℓ is odd and $\langle \iota(\lambda), \check{\theta}_{\sigma,s}\rangle_\sigma$ is an integer, it follows that $\langle \iota(\lambda), \check{\theta}_{\sigma,s}\rangle_\sigma \leq \frac{\ell-1}{2}$.

Note that $P_\sigma = \frac{1}{2}Q_{\sigma,\ell}$ where $Q_{\sigma,\ell}$ is the lattice spanned by long roots of $\mathfrak{g}_\sigma$. Moreover, $\check{h} = 2n+1$ and $\check{h}_\sigma = n+1$ where $\check{h}_\sigma$ is the dual Coxeter number of $\mathfrak{g}_\sigma$. Combining the Verlinde formula (3) and Theorem 1.2, the corollary follows. □

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